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EQUIVALENCE AMONG VISCOELASTIC SPRING-POT GENERAL PROCESSES AND A PURE VISCOUS NEWTONIAN SYSTEM OBSERVED ON A TIME SCALE

[1]LUCIANO BARBANTI, [2]BERENICE DAMASCENO

[1] Assoc. Prof., Department of Mathematics, Fac. of Engineering, Unesp, campus of Ilha Solteira
[2]Asstt Prof., Department of Mathematics, Fac. of Engineering, Unesp, campus of Ilha Solteira
E-mail: [1]barbanti@mat.feis.unesp.br, [2]berenice@mat.feis.unesp.br

ABSTRACT

The use of fractional calculus when modeling phenomena frequently suggests several questions concerning the deepest parts of the physical laws involved in. Here we will be dealing with one instance of such situation. We will be showing that a very large class of viscoelastic systems are equivalent to a pure viscous (Newtonian) process observed in a discrete time scale.

Keywords: ***Spring-pot, fractional derivative, derivative on time scales, Hookean elasticity, Newtonian pure viscosity.***

1. INTRODUCTION

The use of fractional calculus when modeling phenomena proposes queries concerning the laws in Physics.

It is the case, for instance, of the problem on the initial conditions as discussed in [1]. It is also, the case in the work.

This work shows the equivalence between the spring-pots (see [2]) and a pure viscous (Newtonian damper), observed each one considered on different time scales.

In the following we consider as a preparatory matter a little keek on fractional derivatives and on delta derivative in time scales.

2. FRACTIONAL DERIVATIVES:

The theory of derivatives of non-entire (fractional) order goes back to Leibniz in 1695.

In the nineteenth century this theory continued in the works by Riemann, Liouville, Grünwald, Letnikov and during the last century, mainly in the 70's, this new theory blows-up, as long as it was applied in experimental sciences, [3].

The principal applications of fractional differential equations are related with modeling, for instance, diffusion processes in heterogeneous and anisotropic media, in electrochemical processes, viscoelastic fluids, electrical circuits, biological systems, and much others. For these remarks see [4].

There are several definitions of fractional derivative. The most considered fractional order derivatives in the literature are due to Caputo, to Riemann-Liouville and to Grünwald-Letnikov.

Here we will be considering the Caputo fractional order derivatives, because it allows us to be defining integer order initial conditions when dealing with Fractional Order Differential Equations.

Let be α a nonnegative real number with $n-1 < \alpha \leq n\,, n\epsilon\mathbb{N}$, and a mapping $x\colon \mathbb{R} \to \mathbb{R}$.

The Caputo derivative of x is defined, for $t > 0$, as:

$$D^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha+1}x^{(n)}(s)ds$$

The following results are immediate properties of the Caputo derivatives: for every nonnegative real numbers $\alpha\,,\beta$ we have, $D^{\alpha}c = 0$ where c is a constant.

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Looking at the class of monomials we have for $s > 0$:

$$D^{\alpha}\,(t^{s}) = \begin{cases} \frac{\Gamma(s+1)}{\Gamma(s-\alpha+1)} t^{s-\alpha} , & if\ s-\alpha+1>0 \\ 0 & otherwise \end{cases} ,$$

and the semi-group property,

$D^{\alpha+\beta}(t^{s}) = D^{\alpha}(D^{\beta}\,(t^{s}))$.

Notice that it is possible to have the semigroup property in the Caputo derivative for a class substantially more general than the monomials ones.

Namely, the class of functions

$$f \in X_c^p(a,b)\,, \quad c \in \mathbb{R}, \quad 1 \le p \le \infty$$

where $X_c^p(a,b)\,, c \in \mathbb{R}, 1 \le p \le \infty$,is the set of all Lebesgue measurable functions on $[a,b]$ for which

$$\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t}\right)^{\frac{1}{p}} < \infty,$$

$(1 \le p < \infty, c \in \mathbb{R})$,and

$$\|f\|_{X_c^{\infty}} = ess\ sup_{a \le t \le b} |t^c||f(t)|\ , \quad c \in \mathbb{R},$$

(see [5]).

3. TIME SCALES:

Settled by Stefan Hilger in 1988 [6], the Calculus on time scales was created to unify the theories of the continuous time and the discrete dynamical systems. A time scale $\mathbb{T}$ is a closed non-void subset of the real numbers $\mathbb{R}$.

Examples of time scales are:

$\mathbb{R}, \mathbb{Z}\ , \{\frac{1}{2^k}\ ; k \in \mathbb{N}\,\}$,

or the Cantor set.

The fundamental operators in the theory are, for $t \in \mathbb{T}$:

$\sigma(t) = \inf\{r; r > t\}$ - *forward advance* (1)

$\rho(t) = \sup\{r; r < t\}$ - *backward advance*

The $\Delta -$derivative is - for every regulated function f -

$$D_{\mathbb{T}}^{\Delta} f(t) = \begin{cases} f'(t) & if\ t\ is\ dense\ at\ right\ with\ respect\ to\ \mathbb{T} \\ \frac{f(\sigma(t)) - f(t)}{\sigma(t)-t} & if\ t\ is\ isolated\ at\ right\ with\ respect\ to\ \mathbb{T} \end{cases} \quad (2)$$

For more details see [7].

4. LINEAR VISCOELASTICITY:

The equation that describes the behavior of a linear viscoelastic material (due essentially to the Boltzmann superposition rule) is:

$$\sigma(t) = \int_{-\infty}^{t} G(t-s)\varepsilon(s)ds \quad (3)$$

where

σ *is the stress*

ε *is the strain*

and the impulse G, the *force of relaxation*, depending on the model. For details see [8].

If a material stands initially at rest $[that\ is, \varepsilon(t) = 0, for\ t < 0]$ the equation (3) becomes:

$$\sigma(t) = G(t)\varepsilon(0) + \int_0^t G(t-s)\varepsilon(s)ds \quad (4)$$

Then for an **elastic material** (*Hookean case*), the equation in (4) becomes

$$\sigma = E\varepsilon \quad (5)$$

where E is the instantaneous elastic modulus relaxation.

When we are dealing with a **viscous material** (*Newtonian-pure damper case*) we have (4) becoming:

$$\sigma = \mu D^1 \varepsilon = \mu\dot{\varepsilon} \quad (6)$$

where μ is the modulus of viscosity.

Let be the constant, $\tau = \frac{\mu}{E}$ (the relaxation time), and consider the rheological spring-pot (composite) model (see [2]) for $0 \le \alpha \le 1$

$$\sigma = \tau^{\alpha} E D^{\alpha} \varepsilon \quad (7)$$

where D^{α} represents the Caputo fractional derivative of a function.

Notice that in [9] we can find justifications in Physics for the use of (7).

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It is immediate that the equation (7) is a generalization of the equations (5) and (6). In fact:

if $\alpha = 0$ we get (5), the case of the **elastic material** (*Hookean case*),

if $\alpha = 1$ we have (6), the case of the **viscous material** (*Newtonian- pure damper case*).

5. MAIN RESULT:

The spring-pot model in (7), shows that we have two extreme cases in viscoelasticity: the *Hookean case (for* $\alpha = 0$) and the *Newtonian - pure damper case (for* $\alpha = 1$), and all the intermediate situations with $0 < \alpha < 1$.

Notice that at this point with the sake in to be avoiding confusion in notations we will be using until the end of this work, $s(t)$ instead of $\sigma(t)$ for denoting the forward operator in (1).

The main result in this work, shows that all the intermediate cases in a spring-pot model can be interpreted as a Newtonian-pure damper case, but considered on an appropriate discrete time scale $\mathbb{T}$.

That is: for all $0 < \alpha \leq 1$ the spring - pot model (6):

$$\sigma = \tau^{\alpha} E D^{\alpha} \varepsilon$$

can be viewed as a Newtonian pure viscous system,

$$\sigma = \tau^{\alpha} E \frac{d}{dt} \varepsilon$$

defined on a synthesizable time scale $\mathbb{T}$.

In fact: it is sufficient to consider the set of a strictly increasing or strictly decreasing recurrent sequence on $r \in \mathbb{T}$ in which *s(r)* is the solution of

$$\frac{\varepsilon(s(r)) - \varepsilon(r)}{(s(r) - r)} = \frac{\sigma(r)}{\tau^{\alpha} E}. \qquad (8)$$

Observe that in (8) the existence of the implicit function $s(t)$ can be assured by several methods, like the Implicit Function Theorem or different computational methods (see [10]).

As an instance of (8) suppose in the process the strain $\varepsilon(t) = t^p$ and the stress $\sigma(r) = t^q$, $p, q > 0$. Then (8) says that given a point r in some discrete time scale $\mathbb{T}$, we can calculate $s(r)$ according the relation

$$\frac{s(r)^p - r^p}{s(r) - r} = \frac{\sigma(r)}{\tau^{\alpha} E} . \qquad (9)$$

For example: if in (9) we take $p = 2$ and $q = 3$, then

$$s(r) = \frac{r^3}{\tau^{\alpha} E} - r. \qquad (10)$$

Supposing $\tau^{\alpha} E < 1$, then a possible $\mathbb{T}$ is the one in which (10) is fulfilled for every $r \in \mathbb{T}$, beginning at 1, that is:

$$\mathbb{T} = \{1, \frac{1}{\tau^{\alpha} E} - 1, \frac{\left(\frac{1}{\tau^{\alpha} E} - 1\right)^3}{\tau^{\alpha} E} - \tau^{\alpha} E + 1, \ldots\}. \qquad (11)$$

Then, concluding the example: the spring pot

$$t^3 = \tau^{\alpha} E D^{\alpha} t^2 \qquad t \geq 0, \qquad 0 < \alpha \leq 1$$

can be viewed as the pure viscous Newtonian process,

$$t^3 = \tau^{\alpha} E \frac{d}{dt} t^2 = 2\tau^{\alpha} E t$$

defined on the time scale (11).

In general terms the above facts imply the necessity of to be providing an interpretation of the nature of these various processes observed each one in different time scales, in Physics.

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