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PARAMETER ESTIMATION FOR TWO WEIBULL POPULATIONS UNDER JOINT TYPE II CENSORED SCHEME

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ABSTRACT

In this paper, maximum likelihood estimation have been obtained for two Weibull populations under joint type II censored scheme, which generalize results of Balakrishnan and Rasouli (2008). Moreover, approximate confidence region are also discussed and compared with two Bootstrap confidence regions. A numerical illustration for these new results is given.

Keywords: ***Weibull distribution; Joint type-II censoring; Maximum likelihood estimation; Approximate confidence; Bootstrap intervals; Coverage probabilities.***

1. INTRODUCTION

There are various types of censored data to be dealt with in the analysis of lifetime experiments see Lawless (2003). Almost all of these types of data are concerned with the one-sample problems. But, there are situations in which the experimenter plans to compare different populations. In such problems, the joint censoring scheme has been suggested in the literature. As mentioned by Rasouli and Balakrishnan (2010), a joint censoring scheme is quite useful in conducting comparative lifetime test of products coming from different units within the same facility.

More precisely, suppose that the products are being produced by two lines under the same facility. Two independent samples of sizes *m* and *n* are selected from these lines and put simultaneously on a life testing experiment. Then, to save time and money, the experimenter follows a joint censoring scheme and terminates the life testing when a certain number of failures (say, *r*) occur.

Suppose that $X_1,...,X_m$, the lifetimes of *m* specimens of product *A*, are i.i.d. random variables from distribution function $F(x)$ and density function $f(x)$, and $Y_1,...,Y_n$, the lifetimes of *n* specimens of product *B*, are i.i.d. random variables from distribution function *G(x)* and density function *g(x)*. Further, suppose $W_{(1)}<W_{(2)}<...<W_{(N)}$ denote the order statistics of the $N=m+n$ random variables $\{X_1,...,X_m;Y_1,...,Y_n\}$. Then, under the joint type-II censoring scheme, the observable data consist of (Z,W), where $W=(W_{(1)},W_{(2)},...,W_{(r)})$, with r $(1\le r<N)$ being a pre-fixed integer, and $Z=(Z_1,...,Z_r)$ with $z_i=1$ or 0 according as w_i is from an *X*- or *Y*-failure.

Letting $M_r=\sum_{i=1}^{r}Z_i$ denote the number of *X*-failures in W and $N_r=\sum_{i=1}^{r}(1-Z_i)=r-M_r$ (i.e., the number of *Y*-failures in W), the likelihood of *(Z, W)* is given by Balakrishnan and Rasouli (2008) as:

$$L=C\prod_{i=1}^{r}[\{f(w_i)\}^{Z_i}\{g(w_i)\}^{1-Z_i}]\{\bar{F}(w_r)\}^{m-m_r}\{\bar{G}(w_r)\}^{n-n_r}, \quad (1)$$

where $\bar{F}=1-F$, $\bar{G}=1-G$ are the survival functions of the two populations and $C=\frac{m!n!}{(m-m_r)!(n-n_r)!}$.

Balakrishnan and Rasouli (2008) developed likelihood inference for the parameters of two exponential populations under joint type-II censoring. They developed inferential methods based on maximum likelihood estimates (MLE) and compared their performance with those based on some other approaches such as Bootstrap. Shafay et al. (2013) derived the Bayesian inference for the unknown parameters of two exponential populations under joint type II censoring they

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developed with the use of squared-error, linear-exponential and general entropy loss functions. The problem of predicting the future failure times, both point and interval prediction, based on the observed joint type-II censored data is obtained; see also Rasouli and Balakrishnan (2010) for a generalization of their results to progressive type-II censoring. Balakrishnan and Feng (2014) generalized Balakrishnan and Rasouli (2008) and Shafay et al. (2013) works by considered a jointly type II censored sample arising from h independent exponential populations. Finally, Ashour and Abo-Kasem (2014) derived Bayesian and non-Bayesian estimators for two generalized exponential populations under joint type II censored scheme.

Succeeding section deals with the computational procedure to obtain the MLEs of $\beta_1, \beta_2, \theta_1$ and θ_2. The asymptotic variance covariance matrix and approximate confidence region based on the asymptotic normality of the maximum likelihood estimators have been obtained in section 3. While section 4 is describes the various bootstrap confidence intervals. All estimators are not in nice closed forms, therefore, numerical examples are considered to illustrate the proposed estimators in section 5. Last section includes a brief conclusion.

2. MAXIMUM LIKELIHOOD ESTIMATORS

Suppose that the two populations are Weibull distribution with density and distribution functions as

$$f_j(x)=\frac{\beta_j}{\theta_j}\left(\frac{x}{\theta_j}\right)^{\beta_j-1}\exp\left(-\frac{x}{\theta_j}\right)^{\beta_j} \text{ and } F_j(x)=1-\exp\left(-\frac{x}{\theta_j}\right)^{\beta_j},$$

$\beta_j, \theta_j > 0$, $x > 0$, for $j = 1,2$, respectively where β_j are the shape parameters and θ_j are the scale parameters. In this case, the likelihood function in (1) becomes

$$L(\beta_1,\beta_2,\theta_1,\theta_2,w,z)=C\left(\frac{\beta_1}{\theta_1}\right)^{m_r}\left(\frac{\beta_2}{\theta_2}\right)^{n_r} \times\prod_{i=1}^{r}\left[u_i\exp(-u_i)^{\beta_1}\right]^{z_i}\left[q_i\exp(-q_i)^{\beta_2}\right]^{(1-z_i)} \times\left[\exp(v_1)^{\beta_1}\right]^{(m-m_r)}\left[\exp(v_2)^{\beta_2}\right]^{(n-n_r)} \tag{2}$$

where $u_i=\frac{w_i}{\theta_1}, q_i=\frac{w_i}{\theta_2}, v_1=\left(\frac{-w_r}{\theta_1}\right)$ and $v_2=\left(\frac{-w_r}{\theta_2}\right)$.

Therefore, to obtain the MLE's of β_j and θ_j we find the first derivatives of the natural logarithm of the likelihood function (2) with respect to β_j and θ_j and equating them to zero, we get the following four equations

$$\frac{\partial \ln L}{\partial \beta_1}=\frac{m_r}{\hat\beta_1}+\sum_{i=1}^{r}z_i\ln(\hat u_i)-\sum_{i=1}^{r}z_i(\hat u_i)^{\hat\beta_1}\ln(\hat u_i)-(m-m_r)(\hat v_1)^{\hat\beta_1}\ln(\hat v_1)=0,$$

$$\frac{\partial \ln L}{\partial \beta_2}=\frac{n_r}{\hat\beta_2}+\sum_{i=1}^{r}(1-z_i)\ln(\hat q_i)-\sum_{i=1}^{r}(1-z_i)(\hat q_i)^{\hat\beta_2}\ln(\hat q_i)-(n-n_r)(\hat v_2)^{\hat\beta_2}\ln(\hat v_2)=0,$$

$$\frac{\partial \ln L}{\partial \theta_1}=\left(\frac{\hat\beta_1}{\hat\theta_1}\right)\left[-m_r+\left[\sum_{i=1}^{r}z_i(\hat u_i)^{\hat\beta_1}+(m-m_r)(\hat v_1)^{\hat\beta_1}\right]\right]=0,$$

and

$$\frac{\partial \ln L}{\partial \theta_2}=\left(\frac{\hat\beta_2}{\hat\theta_2}\right)\left[-n_r+\left[\sum_{i=1}^{r}(1-z_i)(\hat q_i)^{\hat\beta_2}+(n-n_r)(\hat v_2)^{\hat\beta_2}\right]\right]=0 \tag{3}$$

where $\hat u_i=\left(\frac{w_i}{\hat\theta_1}\right), \hat q_i=\left(\frac{w_i}{\hat\theta_2}\right), \hat v_1=\left(\frac{-w_r}{\hat\theta_1}\right)$ and $\hat v_2=\left(\frac{w_r}{\hat\theta_2}\right)$.

By solving (3), we get the following MLEs of $\hat\beta_j$ and $\hat\theta_j$ for $j=1,2$ as

$$\frac{(m-m_r)(w_r)^{\hat\beta_1}\ln(w_r)+\sum_{i=1}^{r}z_i(w_i)^{\hat\beta_1}\ln(w_i)}{(m-m_r)(w_r)^{\hat\beta_1}+\sum_{i=1}^{r}z_i(w_i)^{\hat\beta_1}}-\frac{1}{\hat\beta_1}=\frac{1}{m_r}\sum_{i=1}^{r}z_i\ln(w_i)$$

and

$$\frac{(n-n_r)(w_r)^{\hat\beta_2}\ln(w_r)+\sum_{i=1}^{r}(1-z_i)(w_i)^{\hat\beta_2}\ln(w_i)}{(n-n_r)(w_r)^{\hat\beta_2}+\sum_{i=1}^{r}(1-z_i)(w_i)^{\hat\beta_2}}-\frac{1}{\hat\beta_2}=\frac{1}{n_r}\sum_{i=1}^{r}(1-z_i)\ln(w_i)$$

which can be solved by using an iterative numerical method, and

$$\hat\theta_1=\left[\frac{(m-m_r)(w_r)^{\hat\beta_1}+\sum_{i=1}^{r}z_i(w_i)^{\hat\beta_1}}{m_r}\right]^{\frac{1}{\hat\beta_1}}$$

and

$$\hat\theta_2=\left[\frac{(n-n_r)(w_r)^{\hat\beta_2}+\sum_{i=1}^{r}(1-z_i)(w_i)^{\hat\beta_2}}{n_r}\right]^{\frac{1}{\hat\beta_2}} \tag{4}$$

Not that for $\beta_j=1$ we obtain MLEs based on a jointly type-II censored sample from two exponential populations which introduced by Balakrishnan and Rasouli (2008).

Remark: From the MLEs in (4), it is evident that when $m_r=\sum_{i=1}^{r}z_i=0$ or r, $\hat\beta_j$ or $\hat\theta_j$ do not exist, respectively. Hence, the MLEs in (4) are only conditional MLEs, conditioned on $1\le m_r\le r-1$.

3. APPROXIMATE CONFIDENCE INTERVALS

The approximate asymptotic variance-covariance matrix for β_j and θ_j for $j=1,2$ can be obtained by inverting the information matrix with the elements that are negative of the expected values of the second order derivatives of logarithms of the likelihood functions. Cohen (1965) concluded that the approximate variance covariance matrix may be obtained by replacing expected values by their MLEs. Now the Fisher information matrix associated with β_j and θ_j is defined as:

$$I(\beta_1,\beta_2,\theta_1,\theta_2)=E\begin{bmatrix} -\frac{\partial^2 \ln L}{\partial \beta_1^2} & 0 & -\frac{\partial^2 \ln L}{\partial \beta_1 \partial \theta_1} & 0 \\ 0 & -\frac{\partial^2 \ln L}{\partial \beta_2^2} & 0 & -\frac{\partial^2 \ln L}{\partial \beta_2 \partial \theta_2} \\ -\frac{\partial^2 \ln L}{\partial \beta_1 \partial \theta_1} & 0 & -\frac{\partial^2 \ln L}{\partial \theta_1^2} & 0 \\ 0 & -\frac{\partial^2 \ln L}{\partial \beta_2 \partial \theta_2} & 0 & -\frac{\partial^2 \ln L}{\partial \theta_2^2} \end{bmatrix},$$

where

$$-\frac{\partial^2 \ln L}{\partial \beta_1^2}=\frac{m_r}{\beta_1^2}+\sum_{i=1}^{r} z_i (u_i)^{\beta_1}\left[\ln(u_i)\right]^2+(m-m_r)(v_1)^{\beta_1}\left[\ln(v_1)\right]^2,$$

$$-\frac{\partial^2 \ln L}{\partial \beta_2^2}=\frac{n_r}{\beta_2^2}+\sum_{i=1}^{r}(1-z_i)(q_i)^{\beta_2}\left[\ln(q_i)\right]^2+(n-n_r)(v_2)^{\beta_2}\left[\ln(v_2)\right]^2,$$

$$-\frac{\partial^2 \ln L}{\partial \theta_1^2}=-m_r\left(\frac{\beta_1}{\theta_1}\right)+\left(\frac{\beta_1}{\theta_1}\right)\left(\frac{\beta_1+1}{\theta_1}\right)\left[\sum_{i=1}^{r} z_i (u_i)^{\beta_1}+(m-m_r)(v_1)^{\beta_1}\right],$$

$$-\frac{\partial^2 \ln L}{\partial \theta_2^2}=-n_r\left(\frac{\beta_2}{\theta_2}\right)+\left(\frac{\beta_2}{\theta_2}\right)\left(\frac{\beta_2+1}{\theta_2}\right)\left[\sum_{i=1}^{r}(1-z_i)(q_i)^{\beta_2}+(n-n_r)(v_2)^{\beta_2}\right],$$

$$-\frac{\partial^2 \ln L}{\partial \beta_1 \partial \theta_1}=\frac{m_r}{\theta_1}-\sum_{i=1}^{r} z_i\left(\frac{1}{\theta_1}\right)(u_i)^{\beta_1}\left[1+\beta_1 \ln(u_i)\right] - (m-m_r)\left(\frac{1}{\theta_1}\right)(v_1)^{\beta_1}\left[1+\beta_1 \ln(v_1)\right],$$

and

$$-\frac{\partial^2 \ln L}{\partial \beta_2 \partial \theta_2}=\frac{n_r}{\theta_2}-\sum_{i=1}^{r}(1-z_i)\left(\frac{1}{\theta_2}\right)(q_i)^{\beta_2}\left[1+\beta_2 \ln(q_i)\right] - (n-n_r)\left(\frac{1}{\theta_2}\right)(v_2)^{\beta_2}\left[1+\beta_2 \ln(v_2)\right] \quad (5)$$

Using the asymptotic normality of the MLEs, we can express the approximate $100(1-\alpha)\%$ confidence intervals for β_j,θ_j for $j=1,2$.

Suppose that $\hat{\delta}$ is the MLE of the parameter vector $\delta=(\beta_1,\beta_2,\theta_1,\theta_2)$. Denote the Fisher information matrix corresponding to δ by I_δ and $\phi=\lim_{n\to\infty} n I_\delta^{-1}$. Then, $\hat{\delta}$ is asymptotically normal distributed (see Serfling (1980)), i.e., $\sqrt{n}(\hat{\delta}-\delta)\sim N(0,\phi)$. In particular, let $\left(\hat{S}_{\hat{\beta}_j}\right)^2=\hat{\phi}_{(j,j)}/n$, $j=1,2$, where $\hat{\phi}_{(j,j)}$, $j=1,2$ are the (j,j) elements in the matrix $\hat{\phi}=n\hat{I}_\delta^{-1}$ and $\hat{I}_\delta$ is the estimator of I_δ. Therefore, asymptotic normality confidence intervals of δ_j, $j=1,2$, with confidence level $100(1-\alpha)\%$ are given by

$$\hat{\beta}_j \pm z_{(1-\alpha/2)}\hat{S}_{\hat{\beta}_j} \text{ and } \hat{\theta}_j \pm z_{(1-\alpha/2)}\hat{S}_{\hat{\theta}_j},\ j=1,2.$$

where $z_{(1-\alpha)/2}$ denotes the upper $(1-\alpha)/2$ percentage point of the standard normal distribution.

Also, an approximate $100(1-\alpha)\%$ simultaneous confidence interval (SCI) for $(\beta_1,\beta_2,\theta_1,\theta_2)$, using the Bonferroni method, can be obtained as

$$\hat{\beta}_j \pm z_{(3+\sqrt{1-\alpha})/4}\hat{S}_{\hat{\beta}_j} \text{ and } \hat{\theta}_j \pm z_{(3+\sqrt{1-\alpha})/4}\hat{S}_{\hat{\theta}_j},\ j=1,2$$

4. BOOTSTRAP INTERVALS

In this section, we present several bootstrap methods to construct confidence intervals for β_j, θ_j for $j=1,2$, viz., Studentized-t interval (Boot-t) and Percentile interval (Boot-p) (see Efron (1982) and Efron and Tibshirani (1994) for details).

a) Bootstrap Percentile Interval Procedure (Boot-p)

The bootstrap percentile method defines the lower and upper bounds of the confidence intervals just using the $100\alpha/2th$ and $100(1-\alpha/2)th$ quantiles of the empirical bootstrap distribution of $\hat{\beta}_j^*$ and $\hat{\theta}_j^*$, $j=1,2$ respectively. In particular:

(1) Compute the MLE $(\hat{\beta}_j,\hat{\theta}_j)$ of (β_j,θ_j) based on two Weibull populations using joint type II censored sample (w,z).

(2) Use $(\hat{\beta}_j,\hat{\theta}_j)$ to generate a bootstrap joint type II censored sample (w^*,z^*) and compute the bootstrap estimate of (β_j,θ_j), say $(\hat{\beta}_j^*,\hat{\theta}_j^*)$, based on this bootstrap sample.

(3) Repeat step 2 B times to have $\hat{\beta}_j^{*(1)},\hat{\beta}_j^{*(2)},\ldots,\hat{\beta}_j^{*(B)}$ and $\hat{\theta}_j^{*(1)},\hat{\theta}_j^{*(2)},\ldots,\hat{\theta}_j^{*(B)}$.

(4) Arrange $\hat{\beta}_j^{*(1)},\hat{\beta}_j^{*(2)},\ldots,\hat{\beta}_j^{*(B)}$ and $\hat{\theta}_j^{*(1)},\hat{\theta}_j^{*(2)},\ldots,\hat{\theta}_j^{*(B)}$ in ascending order and obtain $\hat{\beta}_j^{*[1]},\hat{\beta}_j^{*[2]},\ldots,\hat{\beta}_j^{*[B]}$ and $\hat{\theta}_j^{*[1]},\hat{\theta}_j^{*[2]},\ldots,\hat{\theta}_j^{*[B]}$.

(5) A two-sided $100(1-\alpha)\%$ percentile bootstrap confidence interval for (β_j,θ_j), say $[\hat{\beta}_{jL}^*,\hat{\beta}_{jU}^*]$ and $[\hat{\theta}_{jL}^*,\hat{\theta}_{jU}^*]$ is given by

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$(\hat{\beta}^*_{jL},\hat{\beta}^*_{jU})=\left(\hat{\beta}_j^{*([B\alpha/2])},\hat{\beta}_j^{*([B(1-\alpha/2])}\right)$ and

$$(\hat{\theta}^*_{jL},\hat{\theta}^*_{jU})=\left(\hat{\theta}_j^{*([B\alpha/2])},\hat{\theta}_j^{*([B(1-\alpha/2])}\right)$$

b) Studentized-t Interval Procedure (Boot-t)

The Boot-t confidence intervals estimators are computed according to the following steps:
(1–2) Same as the steps 1–2 in (a).
(3) Compute the *t*–statistic $T_{\hat{\beta}^*_j}=(\hat{\beta}^*_j-\hat{\beta}_j)\big/\hat{S}_{\hat{\beta}^*_j}$ and $T_{\hat{\theta}^*_j}=(\hat{\theta}^*_j-\hat{\theta}_j)\big/\hat{S}_{\hat{\theta}^*_j}$ where $\hat{S}_{\hat{\beta}^*_j}$ and $\hat{S}_{\hat{\theta}^*_j}$ are the bootstrap versions.
(4) Repeat steps 2–3 B times and obtain $T^{(1)}_{\hat{\beta}^*_j},T^{(2)}_{\hat{\beta}^*_j},...,T^{(B)}_{\hat{\beta}^*_j}$ and $T^{(1)}_{\hat{\theta}^*_j},T^{(2)}_{\hat{\theta}^*_j},...,T^{(B)}_{\hat{\theta}^*_j}$.
(5) Arrange $T^{(1)}_{\hat{\beta}^*_j},T^{(2)}_{\hat{\beta}^*_j},...,T^{(B)}_{\hat{\beta}^*_j}$ and $T^{(1)}_{\hat{\theta}^*_j},T^{(2)}_{\hat{\theta}^*_j},...,T^{(B)}_{\hat{\theta}^*_j}$ in ascending order and obtain $T^{[1]}_{\hat{\beta}^*_j},T^{[2]}_{\hat{\beta}^*_j},...,T^{[B]}_{\hat{\beta}^*_j}$ and $T^{[1]}_{\hat{\theta}^*_j},T^{[2]}_{\hat{\theta}^*_j},...,T^{[B]}_{\hat{\theta}^*_j}$.
(6) A two-sided $100(1-\alpha)\%$ bootstrap-*t* confidence interval for (β_j,θ_j) say $[\hat{\beta}^*_{j,tL},\hat{\beta}^*_{j,tU}]$ and $[\hat{\theta}^*_{i,tL},\hat{\theta}^*_{j,tU}]$, is given by

$$\left(\hat{\beta}_j+T^{([B\alpha/2])}_{\hat{\beta}^*_j}\hat{S}_{\hat{\beta}_j},\ \hat{\beta}_j+T^{([B(1-\alpha/2)])}_{\hat{\beta}^*_j}\hat{S}_{\hat{\beta}_j}\right),\ j=1,2,$$

and

$$\left(\hat{\theta}_j+T^{([B\alpha/2])}_{\hat{\theta}^*_j}\hat{S}_{\hat{\theta}_j},\hat{\theta}_j+T^{([B(1-\alpha/2)])}_{\hat{\theta}^*_j}\hat{S}_{\hat{\theta}_j}\right),\ j=1,2,$$

In section 5, we will have a simulation study in order to evaluate the performance of the three confidence intervals.

5. NUMERICAL ILLUSTRATION

It clear that, there are no explicit solutions for obtaining new estimators. Therefore artificial data, numerical solution and computer facilities are needed. The main object of this section is to illustrate numerically most of the new theoretical result obtained in the previous two sections.

5.1 Illustrative Example

In this sub section we present Proschan's Data (1963) which gives failure times (in hours) of the air-conditioning systems of Boeing 720 jet airplanes "7913" and "7914". It is observed that the failure distribution of the air-conditioning system for each of the planes was well approximated by exponential distribution, where m = 24 and n = 27, we ran the various joint censoring schemes on this dataset with r as 20 and 30. The data are presented in table 1.

Table 1: *Failure times of air-conditioning systems in two airplanes*

Plane 7913						
1	4	11	16	18	18	18
24	31	39	46	51	54	63
68	77	80	82	97	106	111
141	142	163	191	206	216	
Plane 7914						
3	5	5	13	14	15	22
22	23	30	36	39	44	46
50	72	79	88	97	102	139
188	197	210				

Table 2 presents the jointly type-II censored data that have been obtained from the two samples in table 1 with r = 20 and 30.

Table 2: *Jointly type-II censored data observed from table 1 with r = 30*

w	1	3	4	5	5	11	13	14	15	16
z	0	1	0	1	1	0	1	1	1	0
w	18	18	18	22	22	23	24	30	31	36
z	0	0	0	1	1	1	0	1	0	1
w	39	39	44	46	46	50	51	54	63	68
z	0	1	1	1	0	1	0	0	0	0

We then computed the MLEs of $(\beta_1,\beta_2,\theta_1,\theta_2)$ and the estimates of their standard deviations for the choices of r = 20 and 30 and these are presented in table 3.

Table 3: *The MLEs and the estimates of their standard deviations based on jointly type-II censored data from table 2.*

r	MLEs $(\hat{\beta}_1,\hat{\beta}_2,\hat{\theta}_1,\hat{\theta}_2)$	$\widehat{SD}$ $(\hat{\beta}_1,\hat{\beta}_2,\hat{\theta}_1,\hat{\theta}_2)$
20	(1.17, 0.96, 54.19, 91.15)	(0.328, 0.306, 16.211, 43.805)
30	(1, 1.02, 65.27, 84.65)	(0.229, 0.241, 17.315, 23.058)

Table 4 presents the 95% approximate, Boot-p and Boot-t confidence intervals for β_1,β_2,θ_1 and θ_2 corresponding to case r = 20 and 30. From these results, we observe that Boot-*p* and Boot-*t* confidence intervals are satisfactory compared to the approximate confidence.

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Table 4: *The 95% approximate, Bootstrap-p and Bootstrap-t confidence intervals for* $\beta_1, \beta_2, \theta_1$ *and* θ_2

	CI for β_1	CI for β_2	CI for θ_1	CI for θ_2
r = 20				
Approximate	(0.529, 1.815)	(0.362, 1.563)	(22.42, 85.97)	(5.294, 177.01)
Boot-*p*	(0.924, 2.56)	(0.682, 4.479)	(18.56, 60.35)	(18.56, 60.35)
Boot-*t*	(0.863, 1,928)	(0.407, 1.887)	(0, 55.262)	(0, 101.139)
r = 30				
Approximate	(0.55, 1.449)	(0.544, 1.487)	(31.34, 99.21)	(39.46, 129.84)
Boot-*p*	(0.83, 1.877)	(0.603, 1.631)	(30.15, 87.97)	(30.147, 87.97)
Boot-*t*	(0.769, 1.573)	(0.396, 1.354)	(0, 101.21)	(49.16, 115.38)

5.2 Monte Carlo simulation

A simulation study was conducted in order to evaluate the performance of MLEs and also all the confidence intervals discussed in the preceding sections. We considered different sample sizes for the two populations as *m* = 15, 20, 30, 50, 80 and *n* = 15, 20, 30, 50, 80 and different choices for *r* = 17, 20, 24, 28, 32, 40, 50, 60, 80, 90, 100, 120,140. We also chose the parameters $(\beta_1, \beta_2, \theta_1, \theta_2)$ to be (4.5, 2.5, 2, 3). For these cases, we computed the MLEs, root mean squared errors $\sqrt{MSE}$ and the 95% approximate confidence intervals for $(\beta_1, \beta_2, \theta_1, \theta_2)$ and the corresponding coverage probabilities. We repeated this process 5000 times and computed the average values of all the estimates. The average value of the MLEs $(\beta_1, \beta_2, \theta_1, \theta_2)$ and ($\sqrt{MSE}$) summarized in table 5.

From table 6 we observe that the coverage probabilities and the average widths of 95% CIs $(\beta_1, \beta_2, \theta_1, \theta_2)$ for approximate confidence intervals are presented for some small, moderate and large values of *m* and *n*.

Table 5*: The average value of the MLEs* $(\beta_1, \beta_2, \theta_1, \theta_2)$ *and (*$\sqrt{MSE}$*) for small, moderate and large values of m , n and r*

(m,n)	r	$\hat{\beta}_1$	$\sqrt{MSE}$	$\hat{\beta}_2$	$\sqrt{MSE}$	$\hat{\theta}_1$	$\sqrt{MSE}$	$\hat{\theta}_2$	$\sqrt{MSE}$
$\beta_1 = 4.5, \beta_2 = 2.5, \theta_1 = 2$ and $\theta_2 = 3$									
	17	5.143	1.671	3.342	2.49	1.986	0.13	3.021	0.953
(15,15)	20	5.008	1.39	3.164	1.68	1.991	0.124	2.955	0.575
	24	4.96	1.252	3.002	1.159	1.992	0.121	2.93	0.39
	24	4.899	1.194	3.047	1.453	1.992	0.111	2.985	0.609
(20,20)	28	4.807	1.011	2.949	1.181	1.995	0.105	2.967	0.44
	32	4.803	0.958	2.866	0.913	1.995	0.104	2.956	0.339
	32	4.816	1.01	2.884	1.11	1.994	0.096	3.015	0.589
(30,30)	40	4.737	0.839	2.802	0.887	1.997	0.088	2.992	0.4
	50	4.718	0.753	2.732	0.637	1.998	0.085	2.974	0.266
	60	4.645	0.659	2.686	0.654	1.996	0.07	3.008	0.377
(50,50)	80	4.625	0.552	2.635	0.473	1.998	0.066	2.981	0.22
	90	4.63	0.548	2.601	0.383	1.998	0.066	2.989	0.188
	100	4.588	0.49	2.613	0.471	1.999	0.054	3.004	0.275
(80,80)	120	4.568	0.422	2.606	0.401	1.9996	0.052	2.988	0.197
	140	4.574	0.41	2.579	0.314	1.9993	0.052	2.993	0.154

Table 6*: Simulated coverage probabilities (CP) and the average widths of the 95% confidence intervals of for some small, moderate and large values of* (n,m) *and r*

(n,m)	r	$\beta_1 = 4.5$		$\beta_2 = 2.5$		$\theta_1 = 2$		$\theta_2 = 3$	
		CP(%)	Length	CP(%)	Length	CP(%)	Length	CP(%)	Length
	17	93.8	5.171	93.04	5.542	93.86	0.49	95.06	3.209
(15,15)	20	93.84	4.469	93.18	4.574	94.12	0.462	95.04	2.219

	24	93.92	4.025	94	3.514	94.04	0.449	94.54	1.464
	24	94.46	4.087	93.78	4.134	94.12	0.42	95.04	2.311
(20,20)	28	94.8	3.583	93.78	3.513	94.54	0.404	95.3	1.741
	32	94.44	3.36	94.04	2.906	94.34	0.396	94.94	1.305
	32	94.22	3.567	94.08	3.423	94.32	0.366	94.8	2.189
(30,30)	40	94.02	2.976	93.96	2.857	94.38	0.334	95.54	1.577
	50	94.54	2.663	94.26	2.134	94.26	0.326	94.14	1.01
	60	94.76	2.444	94.42	2.286	94.62	0.27	95.22	1.426
(50,50)	80	94.82	2.029	94.5	1.697	94.72	0.256	95.48	0.863
	90	94.94	2.011	94.64	1.39	94.78	0.255	95.04	0.727
	100	94.8	1.851	94.44	1.708	95.16	0.211	95.16	1.053
(80,80)	120	94.72	1.623	94.34	1.448	95.24	0.204	95.24	0.778
	140	95	1.568	94.48	1.142	95.18	0.203	94.92	0.596

6. CONCLUSIONS

In this paper, the MLEs for the unknown parameters of two Weibull distributions has been discussed based on a joint type- II censored sample. We obtained the MLEs of the parameters and found corresponding Fisher information matrix. Also, we studied three approximate methods, Asymptotic Normality, Bootstrap-t and parametric Bootstrap percentile procedures for constructing intervals for the parameters. The MLEs have then been compared through a Monte Carlo simulation study and a numerical example has also been presented to illustrate all the inferential results established here. The computational results show that the MLEs have a moderate bias when the essential sample size is small even when the sample sizes *m* and *n* are not small. This bias also seems to affect the approximate confidence intervals based on normality as they are not centered properly in this case. However, the bias of the MLEs becomes negligible when *r* increases, as is evident from. According to the simulation study, when the sample sizes of two populations, *n* and *m*, and the total number of failures *r*, are large, the estimators' biases are small and the confidence intervals have desirable coverage probabilities. Also, we observed that the approximate better than the two bootstrap methods often perform as well as each other.

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