July. 2014. Vol. 5. No. 02 ISSN2305-8269
International Journal of Engineering and Applied Sciences

www.eaas-journal.org

STABILITY OF SOLUTIONS TO THIRD-ORDER NONLINEAR AND NONAUTONOMOUS DIFFERENTIAL EQUATIONS

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Abstract:

We studied the stability and boundedness results of a third-order nonlinear and nonautonomous differential equations of the form $\dddot{x}+\psi(t,x,\dot{x},\ddot{x})\ddot{x}+f(t,x,\dot{x},\ddot{x})=0$. *Generally particular cases of autonomous form and some particular cases of nonautonomous form of this equation have been studied by many authors over the years. However, this particular form is a generalization of the earlier ones. A suitable Lyapunov function was constructed and used for the proof of the main theorem. The results in the paper generalize other authors who have studied particular cases of the differential equations. Finally, a concrete example is given to check our results.*

Keywords: Stability, Third order non-linear, differential equations, Lyapunov function

1. Introduction

We will be considered here with the equations of the form

$$\dddot{x}+\psi(t,x,\dot{x},\ddot{x})\ddot{x}+f(t,x,\dot{x},\ddot{x})=0. \qquad (1)$$

Now, (1) has an equivalent system

$$\begin{aligned}\dot{x}&=y,\\ \dot{y}&=z,\\ \dot{z}&=-\psi(t,x,y,z)z-f(t,x,y,z),\end{aligned} \qquad (2)$$

where $\psi, f\in C([0,\infty)\times R\times R\times R,R)$. We also assume that the real functions ψ, f depend only on the arguments displayed explicitly. The dots denote differentiation with respect to t. Stability, global stability and boundedness of some special cases of (1) have been studied by a number of authors [1-13] over years. Some of these studies have been cited in [12] in detail. Motivation of this study has been based on recent studies of Qian [5], Tunç [8, 10], Omeike [6], and Ateş [12]. Equation (1) is a quite general third-order nonlinear and nonautonomous differential equation. Equations in [1-13] are some special cases of (1), and our study is reducible to the studies in [1-13], but the inversions are not possible for stability case. Thus, the studies which have been done in [1-13] are some special cases of our study for stability case. Hence, our results extend and include those results obtained in [1-13].

2. Preliminaries

Before introducing our main results, we state some basic theorems and a Lyapunov function which will be required in future. Consider the nonautonomous system

$$\dot{x}=f(t,x), \qquad (3)$$

where $f\in C[J\times S_\rho,R^n]$, $J=[t_0,\infty)$, $t_0\ge 0$, and $S_\rho=\{x\in R^n:\|x\|<\rho\}$. Let $f(t,0)=0$ so that (3) admits the zero solution $x(t)\equiv 0$.

Theorem 1. *Assume that there exists a Liapunov function* $V(t,x)$ *such that*

(i) $V\in C[J\times S_\rho,R^+]$, $V(t,0)\equiv 0$, $V(t,x)$ *is positive definite, and* $V(t,x)$ *is*

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locally Lipschitzian in x; *and*

(ii) $\dot{V}(t,x) \le 0, \quad (t,x) \in J \times S_\rho$.

Then, the zero solution $x(t) \equiv 0$ *of* (3) *is stable.*

Proof. See [14].

Theorem 2. *If* f *is differentiable on* X, *then* f *is locally Lipschitzian on* X.

Proof. Let $x \in X$. Since f is differentiable at x,

$$\lim_{y \to x} \frac{\rho\left(f(t,x), f(t,y)\right)}{d(x,y)} = D$$

for some $D \in R$. So for $\varepsilon = 1$, there exists $\delta_1 > 0$ such that if $d(x,y) < \delta_1$ (and $x, y \in X$) then

$$\left| \frac{\rho\left(f(t,x), f(t,y)\right)}{d(x,y)} - D \right| < 1$$

or

$$\left|\rho\left(f(t,x), f(t,y)\right) - Dd(x,y)\right| < \left|d(x,y)\right|$$

and so

$$\left|\rho\left(f(t,x), f(t,y)\right)\right| - \left|Dd(x,y)\right| < \left|d(x,y)\right|$$

and

$$\left|\rho\left(f(t,x), f(t,y)\right)\right| < \left(|D| + 1\right)\left|d(x,y)\right| .$$

Taking $M = |D| + 1$ we see that f is Lipschitzian on the set $B(x, \delta_1)$ and f is locally Lipschitzian on X.

It is well known that the stability is a very important problem in the theory and applications of differential equations. So far, the most effective method to study the stability of nonlinear differential equations is still Lyapunov's second method. The major advantage of this method is that stability in the large can be obtained without any prior knowledge of solutions. Today, this method is widely recognized as an excellent tool not only in the study of differential equations but also in the theory of control systems, dynamical systems, systems with time lag, power system analysis, time-varying nonlinear feedback systems, and so on. Its chief characteristic is the construction of a scalar function, namely, the Lyapunov function. Unfortunately, it is sometimes very difficult to find a proper Lyapunov function for a given system. Therefore, in this work, we construct a suitable Lyapunov function which is an excellent tool in the proof of the main theorems. Here, this function, $V = V(t,x) = V(t,x,y,z)$, is defined by

$$V(t,x,y,z) = \int_0^x f(t,u,0,0)du + \frac{1}{a}\int_0^y f(t,x,v,0)\,dv + \int_0^y \psi(t,x,v,0)vdv + \frac{1}{2a}z^2 + yz \,. \tag{4}$$

Rewrite the function $V(t,x,y,z)$ as follows:

$$\begin{aligned} V(t,x,y,z) &= \frac{1}{2a}(ay+z)^2 + \frac{1}{2ab}[f(t,x,0,0)+by]^2 + \int_0^y [\psi(t,x,v,0)-a]vdv \\ &\quad + \frac{1}{a}\int_0^y [f_v(t,x,\theta_1 v,0)-b]vdv + \int_0^x [1-\frac{1}{ab}f_u(t,u,0,0)]f(t,u,0,0)du, \end{aligned} \tag{5}$$

where

$$f_v(t,x,\theta_1 v,0) = \frac{f(t,x,v,0)-f(t,x,0,0)}{v}, \; (v \neq 0, \;\; 0 \le \theta_1 \le 1). \tag{6}$$

3. Main results

We have the following.

Theorem 3. *Suppose there exist positive constants* $t \ge 0, \;\; \delta_0$ *(sufficiently small),* a, b, c

such that $ab > c$, *and the following conditions are satisfied:*

(i) $f(t,x,0,0)/x \ge \delta_0 \quad (x \ne 0)$,

(ii) $f_x(t,x,0,0) \le c$,

(iii) $\psi(t,x,y,z) \ge a$,

(iv) $f_y(t,x,y,0) \ge b$ *for all* x, y *and* z,

(v) $y\psi_z(t,x,y,z) \ge 0,\ f_z(t,x,y,z) \ge 0$ *for all* x, y *and* z,

(vi) $\int_0^x f_t(t,u,0,0)du + \frac{1}{a}\int_0^y f_t(t,x,v,0)dv + \int_0^y \psi_t(t,x,v,0)\, v dv \le 0,$

(vii) $a\left[f(t,x,y,z) - f(t,x,0,0) - \int_0^y \psi_x(t,x,v,0) v dv \right] y \ge y \int_0^y f_x(t,x,v,0)dv.$

Then, the zero solution $x(t) \equiv 0$ *of* (1) *is stable.*

Proof. From conditions (i)-(iv) of Theorem 3, we obtain

$$V \ge \frac{1}{2a}(ay+z)^2 + \frac{1}{2ab}\left[f(t,x,0,0)+by\right]^2 + \frac{1}{2}\delta_1 x^2 \tag{7}$$

where $\delta_1 = \frac{1}{ab}(ab-c)\delta_0 > 0$. It follows that there exists a constant $D_0 > 0$ small enough that

$$V(t,x,y,z) \ge D_0(x^2+y^2+z^2). \tag{8}$$

Hence, V is positive definite.

Now, we show that the derivative of V with respect to t along the solution path of system (2) is negative semidefinite.

Let

$$\dot{V} \equiv -U, \tag{9}$$

where

$$U = \left[\frac{\psi(t,x,y,z)-\psi(t,x,y,0)}{z} y + \frac{1}{a}\frac{f(t,x,y,z)-f(t,x,y,0)}{z}\right] z^2 + \left[\frac{1}{a}\psi(t,x,y,z) - 1\right] z^2$$
$$+ y\left[f(t,x,y,z) - f(t,x,0,0) - \int_0^y \psi_x(t,x,v,0) v dv - \frac{1}{a}\int_0^y f_x(t,x,v,0)dv \right] \tag{10}$$
$$- \int_0^x f_t(t,u,0,0)du - \frac{1}{a}\int_0^y f_t(t,x,v,0)dv - \int_0^y \psi_t(t,x,v,0) v dv.$$

First, from condition (v) of Theorem 3, we obtain

$$\left[\frac{\psi(t,x,y,z)-\psi(t,x,y,0)}{z}\right] yz^2 = y\psi_z(t,x,y,\theta_2 z)z^2 \ge 0,\ 0 \le \theta_2 \le 1,$$
$$\left[\frac{f(t,x,y,z)-f(t,x,y,0)}{z}\right] z^2 = f_z(t,x,y,\theta_3 z)z^2 \ge 0,\ 0 \le \theta_3 \le 1. \tag{11}$$

Next, from conditions (iii), (v), (vi) and (vii) Theorem 3, we obtain that $U \ge 0$.

Hence

$$\dot{V}_{(2)}(t,x,y,z) \le 0. \tag{12}$$

The whole discussions (conditions of Theorems 1, 2 and 3) show that the zero solution of system (2) is stable. Then, the rest of the proof may now follow as in [2].

Thus, the proof of Theorem 3 is completed.

4. Example

Consider the equation

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$$\dddot{x} + \left[(\sin x)\dot{x} + (\dot{x})^2 + e^{\dot{x}\ddot{x}} + e^{-t} + 4\right]\ddot{x} + (\dot{x})^3 + \dot{x} + \frac{x}{1+x^2} + \dot{x}e^{\dot{x}\ddot{x}} + \dot{x}e^{-at} = \frac{1}{1+t^2+x+\dot{x}+\ddot{x}}. \tag{13}$$

Equation (13) is in the form of (1), where

$$\psi(t,x,y,z) = (\sin x)y + y^2 + e^{yz} + e^{-t} + 4,$$
$$f(t,x,y,z) = y^3 + y + \frac{x}{1+x^2} + ye^{yz} + ye^{-at}, \tag{14}$$
$$P(t,x,y,z) = \frac{1}{1+t^2+x^2+y^2+z^2}.$$

With $a=2,\ b=1,\ c=1,$ from conditions (vi) and (vii) of Theorem 3, we have

$$-(e^{-t} + e^{-2t}) \le 0,$$
$$2\left(\frac{2}{3}y^2 + e^{yz} + e^{-2t} + 1\right) \ge \frac{1-x^2}{(1+x^2)^2}, \tag{15}$$

respectively.
Hence, conditions (vi) and (vii) of Theorem 3 are satisfied.

Then, it is easy to check that all the other conditions [(i)-(v)] of Theorem 3 are satisfied. Hence, the trivial solution of (13) is stable.

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