June. 2014. Vol. 5. No. 01 ISSN2305-8269
International Journal of Engineering and Applied Sciences

www.eaas-journal.org

SOME CHARACTERIZATIONS FOR THE EXISTINCE OF BERTRAND CURVES IN DUAL SPACE D^4

Emel Karaca*, Mustafa Çalışkan, Sevinç Aydın*****
Department of Mathematics, Faculty of Science, University of Gazi, Ankara, Turkey
E-mail: emelkkaraca@gmail.com*,mustafacaliskan@gazi.edu.tr***, sevinccaydin@gmail.com***

ABSTRACT

In differential geometry, the characterizations of curves and corresponding relations between the curves are an important problem. Bertrand curves can be given as example of that relation. A Bertrand curve in IR^3 is a curve such that its principal normal vectors are the principal normal vectors of an other curve. Aminov proved that a Bertrand curve does not exist in IR^n if $n \geq 4$. Kim, Park and Yorozu provided some examples illustrating the resultant curves. In this study, generally, we proved and expressed that Bertrand curves are absent in ID^4, but we expressed that Bertrand curves can be found under some characterizations and special conditions in ID^4.

Keywords: *Dual Frenet frames, Bertrand curves, Dual space.*

I.INTRODUCTION AND PRELIMINARIES

We recall some basic notions about dual space and apparatus of curves.

The set $(ID^3, +, .)$ is a module on the ring ID which is called ID-Module and the elements are dual vectors consisting of two real vectors. In ID^3, each dual vector is shown as $\vec{A} = \vec{a} + \varepsilon\vec{a}^*$ and $\varepsilon^2 = 0$ which is called a dual unit.

The set $\{ \vec{X} = \vec{x} + \varepsilon\vec{x}^* : \|\vec{X}\| = (1, 0), \vec{x}, \vec{x}^* \in ID^3\}$ is a unit dual sphere in ID^3.

For $\vec{A} = \vec{a} + \varepsilon\vec{a}^*$, if $\vec{a} \neq 0$ and $\vec{a}^* \neq 0$, $\vec{A}$ is called unique dual vector.

The inner product of $\vec{A} = \vec{a} + \varepsilon\vec{a}^*$ and $\vec{B} = \vec{b} + \varepsilon\vec{b}^*$ is defined by

$$f : ID^3 \times ID^3 \longrightarrow ID^3$$

$$(\vec{A}, \vec{B}) \longrightarrow f(\vec{A}, \vec{B}) = <\vec{A}, \vec{B}>$$

$$= <\vec{a} + \varepsilon\vec{a}^*, \vec{b} + \varepsilon\vec{b}^*>$$

$$= <\vec{a}, \vec{b}> + \varepsilon [<\vec{a}^*, \vec{b}> + <\vec{a}, \vec{b}^*>.$$

Let $\vec{A}, \vec{B}, \vec{C}$ be unique dual vectors in ID^3 and $\lambda_i = c_i + \varepsilon c_i * \in ID$ $(1 \leq i \leq 3)$ be dual number. If $\lambda_1\vec{A} + \lambda_2\vec{B} + \lambda_3\vec{C} = 0$ is provided for $\forall\lambda_i = 0$, the unique dual vectors $\vec{A}, \vec{B}, \vec{C}$ are linear independent.

Let $\vec{A}, \vec{B}, \vec{C}$ be dual vectors in ID^3 and $\lambda_i = c_i + \varepsilon c_i*$ $\in ID$ $(1 \leq i \leq 3)$ be dual number. If $\lambda_1\vec{A} + \lambda_2\vec{B} + \lambda_3\vec{C} = 0$ is provided for $\exists\lambda_i \neq 0$, the dual vectors $\vec{A}, \vec{B}, \vec{C}$ are linear dependent.

Let ID^3 be the dual space with standart inner product $<,>$ and $\vec{\alpha}$ and $\vec{\beta}$ be the dual space curves. If there exists a corresponding relationship between the dual space curves $\vec{\alpha}$ and $\vec{\beta}$ so that the principal normal vectors of $\vec{\alpha}$ and $\vec{\beta}$ are linear dependent to each other at the corresponding points of the dual curves, then $\vec{\alpha}$ and $\vec{\beta}$ are called Bertrand curves in ID^3.

In the following theorems, the characterizations of a dual Bertrand curve are obtained.

Let $\vec{\alpha}$ and $\vec{\beta}$ be two curves in ID^3. If $\vec{\alpha}$ and $\vec{\beta}$ are Bertrand curves, then

$$d\,(\vec{\alpha}(s), \vec{\beta}(s)) = \lambda$$

where $s \in I \subset ID$ and $\lambda \in ID$ (constant).

Let $\vec{\alpha}$ and $\vec{\beta}$ be two curves in ID^3. If $\vec{\alpha}$ and $\vec{\beta}$ are Bertrand curves, then the dual angle between the tangent vectors at the corresponding points of the dual Bertrand curves is constant.

 ISSN2305-8269
International Journal of Engineering and Applied Sciences

www.eaas-journal.org

Let $\vec{\alpha}$ and $\vec{\beta}$ be two curves in ID^3. If $\vec{\alpha}$ and $\vec{\beta}$ are Bertrand curves, κ(s) and τ(s) are the curvature and torsion of $\vec{\alpha}$, κ^0(s) and τ^0(s) are the curvature and torsion of $\vec{\beta}$, respectively, then,

$$\lambda\, \kappa(s) + \mu\, \tau(s) = 1$$

where $\lambda, \mu \in ID$ are constant.

II. SOME CHARACTERIZATIONS FOR THE EXISTINCE OF BERTRAND CURVES IN DUAL SPACE ID^4

In this section, we obtain some characterizations in ID^4. Then, we give the main theorem for existence of Bertrand curves under special conditions in ID^4.

Definition II.1. The set $(ID^4, +, .)$ is a module on the ring ID . We show that

$$ID^4 = \{(A_1, A_2, A_3, A_4) : A_1, A_2, A_3, A_4 \in ID\}.$$

Definition II.2. The inner product of $\vec{A} = \vec{a} + \varepsilon\vec{a}^*$ and $\vec{B} = \vec{b} + \varepsilon\vec{b}^*$ is defined by

$$f : ID^4 \times ID^4 \longrightarrow ID^4$$

$$(\vec{A}, \vec{B}) \longrightarrow f(\vec{A}, \vec{B}) = < \vec{A}, \vec{B} >$$

$$= < \vec{a} + \varepsilon\vec{a}^*, \vec{b} + \varepsilon\vec{b}^* >$$

$$= < \vec{a}, \vec{b} > + \varepsilon\, [< \vec{a}^*, \vec{b} > + < \vec{a}, \vec{b}^* >.$$

Definition II.3. Let $\vec{A}, \vec{B}, \vec{C}, \vec{D}$ be unique dual vectors in ID^4 and $\lambda_i = c_i + \varepsilon c_i^* \in ID$ $(1 \le i \le 4)$ be dual number. If $\lambda_1\vec{A} + \lambda_2\vec{B} + \lambda_3\vec{C} + \lambda_4\vec{D} = 0$ is provided for $\forall\lambda_i = 0$, the unique dual vectors $\vec{A}, \vec{B}, \vec{C}, \vec{D}$ are linear independent.

Definition II.4. Let $\vec{A}, \vec{B}, \vec{C}, \vec{D}$ be dual vectors in ID^4 and $\lambda_i = c_i + \varepsilon c_i^* \in ID$ $(1 \le i \le 4)$ be dual number. If $\lambda_1\vec{A} + \lambda_2\vec{B} + \lambda_3\vec{C} + \lambda_4\vec{D} = 0$ is provided for $\exists\lambda_i \ne 0$, the dual vectors $\vec{A}, \vec{B}, \vec{C}, \vec{D}$ are linear dependent.

Let the curves $\vec{\alpha}$ and $\vec{\beta}$ be Bertrand curves in ID^4. Let $\{\vec{T}, \vec{N}, \vec{B}\}$ indicate the Frenet vectors along $\vec{\alpha}$ and $\{\vec{T}^0, \vec{N}^0, \vec{B}^0\}$ the Frenet vectors along $\vec{\beta}$.

For the dual curve $\vec{\alpha}$,

$$\vec{T} = \vec{t}_1 + \varepsilon\vec{t}_1^*,$$

$$\vec{N} = \vec{n}_1 + \varepsilon\vec{n}_1^*,$$

$$\vec{B} = \vec{b}_1 + \varepsilon\vec{b}_1^*,$$

$$\kappa = k_1 + \varepsilon k_1^*,$$

$$\tau = k_2 + \varepsilon k_2^*.$$

For the dual curve $\vec{\beta}$,

$$\vec{T}^0 = \vec{t}_2 + \varepsilon\vec{t}_2^*,$$

$$\vec{N}^0 = \vec{n}_2 + \varepsilon\vec{n}_2^*,$$

$$\vec{B}^0 = \vec{b}_2 + \varepsilon\vec{b}_2^*,$$

$$\kappa^0 = k_1{}^0 + k_1{}^0*,$$

$$\tau^0 = k_2{}^0 + k_2{}^0*.$$

Theorem II.1. The curves whose principal normal vectors are pure dual vectors are Bertrand curves on the unit dual sphere in ID^4.

Proof. Let $\vec{\alpha}$ and $\vec{\beta}$ be two curves in ID^4. Respectively, $\vec{N}$ and $\vec{N}^0$ are principal normal vectors of $\vec{\alpha}$ and $\vec{\beta}$. If $\vec{\alpha}$ and $\vec{\beta}$ are Bertrand curves in ID^4, the principal normal vectors of $\vec{\alpha}$ and $\vec{\beta}$ are linear dependent. In other words, $\lambda_1\vec{N} + \lambda_2\overrightarrow{N^0} = 0$ is provided for $\exists\lambda_i \ne 0$. Then,

$\lambda_1(\vec{n}_1 + \varepsilon\vec{n}_1^*) + \lambda_2(\vec{n}_2 + \varepsilon\vec{n}_2^*) = 0$ is linear dependent for $\exists\lambda_i \ne 0$ (?)

$$(c_1 + \varepsilon c_1^*)(\vec{n}_1 + \varepsilon\vec{n}_1^*) + (c_2 + \varepsilon c_2^*)(\vec{n}_2 + \varepsilon\vec{n}_2^*) = 0,$$

$$c_1\, \vec{n}_1 + c_2\, \vec{n}_2 + \varepsilon\, (c_1^*\vec{n}_1 + \vec{n}_1^* c_1 + c_2^*\vec{n}_2 + c_2\, \vec{n}_2^*) = 0.$$

Then,

$$c_1\, \vec{n}_1 + c_2\, \vec{n}_2 = 0, \tag{1}$$

$$c_1^*\vec{n}_1 + \vec{n}_1^* c_1 + c_2^*\vec{n}_2 + c_2\, \vec{n}_2^* = 0. \tag{2}$$

From (1), we get

$$c_1\, \vec{n}_1 = -\, c_2\, \vec{n}_2\, ,$$

$$\vec{n}_1 = -\frac{c_2}{c_1}\vec{n}_2\, . \tag{3}$$

Specially, we accept $c_1 \ne 0$. Otherwise, the expression is undefined. If we take $-\frac{c_2}{c_1} = k$, the equation (3) is $\vec{n}_1 = k\vec{n}_2$.

If $\vec{n}_2 = 0$ is accepted, $\vec{n}_1 = 0$. The equation (2)

$$\vec{n}_1^* c_1 + c_2\, \vec{n}_2^* = 0. \tag{4}$$

From (4), we get

$$\vec{n}_1^* c_1 = -c_2\, \vec{n}_2^*,$$

International Journal of Engineering and Applied Sciences

www.eaas-journal.org

$$\vec{n}_1{}^* = -\frac{c_2}{c_1}\vec{n}_2{}^*,$$

$$\vec{n}_1{}^* = k\,\vec{n}_2{}^*.$$

Finally, we say that the principal normal vectors of $\vec{\alpha}$ and $\vec{\beta}$ are linear dependent.

In proof, if $\vec{n}_2 \neq 0$ is accepted, two dual curves are Bertrand curves under special conditions. We give two conclusions about these situations.

Corollary I. If λ_1 and λ_2 are obtained pure reel, two dual curves are Bertrand curves.

Corollary II. The dual curves $\vec{\alpha}$ and $\vec{\beta}$ are unique dual vectors, there is no curve whose principal normal vectors are linear dependent.

References

1. [1] C.Y. Kim, J. Park, S. Yorozu, Curves on the unit 3-sphere in Euclidean 4-space, Bull, Korean Math. Soc., No. 5 (2013) 1599-1622.
2. [2] E. Karaca, M. Çalışkan, Some characterizations for geodesic sprays in dual space, International Journal of Engineering and Applied Sciences, Vol.4, No.10 (2014).
3. [3] H. H. Hacısalihoğlu, Hareket geometrisi ve kuaterniyonlar teorisi, Gazi Üniversitesi Fen-Edebiyat Fakültesi (1983).
4. [4] H. H. Hacısalihoğlu, Diferensiyel geometri, Cilt1-2, Ankara Üniversitesi Fen Fakültesi (2000).
5. [5] İ. A. Güven, İ. Ağaoğlu, The properties of Bertrand curves in dual space, Cornell University Library (2013).
6. [6] Yu. Aminov, Differential geometry and topology of curves, Gordon and Breach Science Publishers, Amsterdam (2000).