Feb, 2014. Vol. 4, No. 9 ISSN2305-8269
International Journal of Engineering and Applied Sciences

www.eaas-journal.org

ON APPROXIMATE SOLUTIONS FOR FRACTIONAL RICCATI DI ERENTIAL EQUATION

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Abstract

In the present study, a new approximate formula of the fractional derivatives is derived. The proposed formula is based on the generalized Laguerre polynomials. Global approximations to functions de ned on a semi-in nite interval are constructed. The fractional derivatives are presented in terms of Caputo sense. Special attention is given to study the convergence analysis and estimate an error upper bound of the presented formula. The new spectral Laguerre collocation method is presented for solving fractional Riccati di erential equation (FRDE). The properties of Laguerre polynomials approximation are used to reduce FRDE to solve a system of algebraic equations which solved using a suitable numerical method. Numerical results are provided to con rm the theoretical results and the e ciency of the proposed method.

Keywords: Fractional Riccati di erential equation; Caputo fractional derivative; Laguerre polynomials; Laguerre-spectral collocation method; Convergence analysis.

1. Introduction

Ordinary and partial fractional di erential equations (FDEs) have been the focus of many studies due to their frequent appearance in various applications in uid mechanics, viscoelas-ticity, biology, physics and engineering [20]. Fractional calculus is a generalization of ordinary di erentiation and integration to an arbitrary non-integer order. Many physical processes appear to exhibit fractional order behavior that may vary with time or space. Most FDEs do not have exact solutions, so approximate and numerical techniques ([13], [24]), must be used. Several numerical and approximate methods to solve FDEs have been given such as variational iter-ation method [23], homotopy perturbation method [12], Adomian decomposition method [19], homotopy analysis method [27] and collocation method ([9], [11], [13], [25]).

The Riccati di erential equation is named after the Italian Nobleman Count Jacopo Francesco Riccati (1676-1754). The book of Reid [21] contains the fundamental theories of Riccati equa-tion, with applications to random processes, optimal control, and di usion problems. Besides important engineering science applications that today are considered classical, such as stochastic realization theory, optimal control, robust stabilization, and network synthesis, the newer appli-cations include such areas as nancial mathematics [16]. The solution of this equation can be reached using classical numerical methods such as the forward Euler method and Runge-Kutta method. An unconditionally stable scheme was presented by Dubois and Saidi [5]. Bahnasawi et al. [3] presented the usage of Adomian decomposition method to solve the non-linear Riccati di erential equation in an analytic form. Tan and Abbasbandy [27] employed the analytic tech-nique called homotopy analysis method to solve the quadratic Riccati equation.

The fractional Riccati di erential equation is studied by many authors and using di erent nu-merical methods. This problem is solved using by variational iteration method [7] and in [19] it solved using the Adomian decomposition method and others [14].

 ISSN2305-8269
International Journal of Engineering and Applied Sciences

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Representation of a function in terms of a series expansion using orthogonal polynomials is a fundamental concept in approximation theory and forms the basis of spectral methods of solution of di erential equations [15]. In [9], Khader introduced an e cient numerical method for solving the fractional di usion equation using the shifted Chebyshev polynomials. In [4] the generalized Laguerre polynomials were used to compute a spectral solution of a non-linear boundary value problems.

The generalized Laguerre polynomials constitute a complete orthogonal sets of functions on the semi-in nite interval [0; 1). Convolution structures of Laguerre polynomials were presented in [2]. Also, other spectral methods based on other orthogonal polynomials are used to obtain spectral solutions on unbounded intervals [28]. Spectral collocation methods are e cient and highly accurate techniques for numerical solution of non-linear di erential equations. The basic idea of the spectral collocation method is to assume that the unknown solution u(t) can be approximated by a linear combination of some basis functions, called the trial functions, such as orthogonal polynomials. The orthogonal polynomials can be chosen according to their special properties, which make them particularly suitable for a problem under consideration.

The main aims of the presented paper is concerned with an extension of the previous work on fractional di erential equations and derive an approximate formula of the fractional derivative of the Laguerre polynomials and then we applied this approach to obtain the numerical solution of FRDE. Also, we presented study of the convergence analysis of the proposed method.

In this article, we consider the fractional Riccati di erential equation of the form

$$D^{\nu}u(t)+u^2(t)-1 = 0, \ t > 0, \ 0 < \nu \leq 1, \qquad (1)$$

the parameter ν refers to the fractional order of the time derivative. We also assume an initial condition

$$u(0) = u^{0.} \qquad (2)$$

For ν = 1; Eq.(1) is the standard Riccati di erential equation

$$\frac{du(t)}{dt} + u^2(t) - 1 = 0.$$

The exact solution to this equation is

$$u(t) = \frac{e^{2t}-1}{e^{2t}+1}.$$

The structure of this paper is arranged in the following way: In section 2, we introduce some basic de nitions about Caputo fractional derivatives and properties the Laguerre polynomials. Section 3, is assigned to study the existence and the uniqueness of the fractional Riccati di er-ential equation. In section 4, an approximate formula of the fractional derivative of Laguerre polynomials and its convergence analysis are given. In section 5, the numerical implementation of the proposed method is given for solving FRDE to show the accuracy of the presented method. Finally, in section 6, the report ends with a brief conclusion and some remarks.

2. Preliminaries and notations

In this section, we present some necessary de nitions and mathematical preliminaries of the fractional calculus theory required for our subsequent development.

2.1 The Caputo fractional derivative

Definition 1.

The Caputo fractional derivative operator D of order is de ned in the following form

$$D^{\nu}f(x) = \frac{1}{\Gamma(m-\nu)}\int_0^x \frac{f^{(m)}(t)}{(x-t)^{\nu-m+1}}dt, \quad \nu > 0, \quad x > 0,$$

where $m-1 < \nu \leq m$, $m \in N$.

Similar to integer-order di erentiation, Caputo fractional derivative operator is a linear operation

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$$D^{\nu}(\lambda f(x) + \mu g(x)) = \lambda D^{\nu} f(x) + \mu D^{\nu} g(x),$$

where λ and μ are constants. For the Caputo's derivative we have

$$D^{\nu} C = 0, \qquad C \text{ is a constant}, \qquad (3)$$

$$D^{\nu} x^{n} = \begin{cases} 0, & \text{for } n \in \mathbb{N}_0 \text{ and } n < \lceil \nu \rceil; \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\nu)} x^{n-\nu}, & \text{for } n \in \mathbb{N}_0 \text{ and } n \geq \lceil \nu \rceil. \end{cases}$$

We use the ceiling function $\lceil \nu \rceil$ to denote the smallest integer greater than or equal to ν, and N0 ={0,1,2,...}. Recall that for ν ∈N, the Caputo differential operator coincides with the usual differential operator of integer order. For more details on fractional derivatives definitions and its properties see ([20], [22]).

2.2 The definition and properties of the classical Laguerre polynomials

The classical Laguerre polynomials $[L_n^{(\alpha)}(x)]_{n=0}^{\infty}, \ \alpha > -1$ are defined on the unbounded interval (0,∞) and can be determined with the aid of the following recurrence formula

$$(n+1)L_{n+1}^{(\alpha)}(x) + (x - 2n - \alpha - 1)L_n^{(\alpha)}(x) + (n+\alpha)L_{n-1}^{(\alpha)}(x) = 0, \qquad n = 1, 2, 3, ..., \qquad (5)$$

where, $L^{(\alpha)}_0(x) = 1$ and $L^{(\alpha)}_1(x) = \alpha+1-x$.

The analytic form of these polynomials of degree n is given by

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n+\alpha}{n-k} x^k = \binom{n+\alpha}{n} \sum_{k=0}^{n} \frac{(-n)_k}{(\alpha+1)_k} \frac{x^k}{k!}, \qquad (6)$$

$L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}$. These polynomials are orthogonal on the interval [0,∞) with respect to the weight function $w(x) = \frac{1}{\Gamma(1+\alpha)} x^{\alpha} e^{-x}$. The orthogonality relation is

$$\frac{1}{\Gamma(1+\alpha)} \int_0^{\infty} x^{\alpha} e^{-x} L_m^{(\alpha)}(x) L_n^{(\alpha)}(x) dx = \binom{n+\alpha}{n} \delta_{mn}. \qquad (7)$$

Also, they satisfy the differentiation formula

$$D^k L_n^{(\alpha)}(x) = (-1)^k L_{n-k}^{(\alpha+k)}(x), \qquad k = 0, 1, ..., n. \qquad (8)$$

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Any functionu(x) belongs to the space $L^2{}_w[0,\infty)$ of all square integrable functions on $[0,\infty)$with weight function w(x), can be expanded in the following Laguerre series

$$u(x) = \sum_{i=0}^{\infty} c_i L_i^{(\alpha)}(x), \tag{9}$$

where the coefficients ci are given by

$$c_i = \frac{\Gamma(i+1)}{\Gamma(i+\alpha+1)} \int_0^{\infty} x^{\alpha} e^{-x} L_i^{(\alpha)}(x) u(x) dx, \quad i = 0, 1, \dots . \tag{10}$$

Consider only the first (m+1) terms of classical Laguerre polynomials, so we can write

$$u_m(x) = \sum_{i=0}^{m} c_i L_i^{(\alpha)}(x). \tag{11}$$

For more details on Laguerre polynomials, its definitions and properties see ([4], [26], [28]).

3. Existence and uniqueness

Let J = [0,T], T < ∞ and C(J) be the class of all continuous functions defined on J, with the norm

$$||u|| = \sup_{t \in J} |e^{-Nt} u(t)|, \quad N > 0,$$

which is equivalent to the sup-norm $||u|| = \sup_{t \in J} |u(t)|.$

To study the existence and the uniqueness of the initial value problem of the fractional Riccati differential equation (1), we suppose that the solution u(t) belongs to the space B = {u ∈ R : |u| ≤ b}, for some constant b.

Theorem 1. ([6], [11])

The initial value problem (1) has a unique solution

$$u \in C(J), \; u' \in X = \{u \in L_1[0,T], \quad ||u|| = ||e^{-Nt} u(t)||_{L_1}\}.$$

4. The approximate fractional derivatives of L(α) n (x) and its convergence analysis

The main goal of this section is to introduce the following theorems to derive an approximate formula of the fractional derivatives of the classical Laguerre polynomials and study the truncating error and its convergence analysis.

The main approximate formula of the fractional derivative of u(x) is given in the following theorem.

Theorem 2.

Let u(x) be approximated by the generalized Laguerre polynomials as (11) and also suppose ν > 0 then, its approximated fractional derivative can be written in the following form

$$D^{\nu}(u_m(x)) \cong \sum_{i=\lceil \nu \rceil}^{m} \sum_{k=\lceil \nu \rceil}^{i} c_i \, w_{i,k}^{(\nu)} \, x^{k-\nu}, \tag{12}$$

where $w_{i,k}^{(\nu)}$ is given by

$$w_{i,k}^{(\nu)} = \frac{(-1)^k}{\Gamma(k+1-\nu)} \begin{pmatrix} i+\alpha \\ i-k \end{pmatrix}. \tag{13}$$

Proof. Since the Caputo's fractional differentiation is a linear operation, then from (6) we have

$$D^{\nu}(u_m(x)) = \sum_{i=0}^{m} c_i \, D^{\nu}(L_i^{(\alpha)}(x)). \tag{14}$$

Since

$$D^{\nu} L_i^{(\alpha)}(x) = 0, \qquad i = 0, 1, ..., \lceil \nu \rceil - 1, \quad \nu > 0. \tag{15}$$

Therefore, for i = • ν • ,...,m, and by using Eqs.(3)-(4) in Eq.(6), we get

$$D^{\nu} L_i^{(\alpha)}(x) = \sum_{k=0}^{i} \frac{(-1)^k}{k!} \begin{pmatrix} i+\alpha \\ i-k \end{pmatrix} D^{\nu} x^k = \sum_{k=\lceil \nu \rceil}^{i} \frac{(-1)^k}{\Gamma(k+1-\nu)} \begin{pmatrix} i+\alpha \\ i-k \end{pmatrix} x^{k-\nu}. \tag{16}$$

A combination of Eqs.(14)-(16) leads to the desired result.

Theorem3.

The Caputo fractional derivative of order ν for the generalized classical Laguerre polynomials can be expressed in terms of the generalized Laguerre polynomials themselves in the following form

$$D^{\nu} L_i^{(\alpha)}(x) = \sum_{k=\lceil \nu \rceil}^{i} \sum_{j=0}^{k-\lceil \nu \rceil} \Omega_{i\,j\,k} \, L_j^{(\alpha)}(x), \qquad i = \lceil \nu \rceil, \lceil \nu \rceil + 1, ..., m, \tag{17}$$

where

$$\Omega_{i\,j\,k} = \frac{(-1)^{j+k} \, (\alpha+i)! \, (k-\nu+\alpha)!}{(i-k)! \, (\alpha+k)! \, (k-\nu-j)! \, (\alpha+j)!}.$$

Proof. From the properties of the generalized Laguerre polynomials [1] and expand xk−ν in Eq.(16) in the following form

ISSN2305-8269
International Journal of Engineering and Applied Sciences

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$$x^{k-\nu} = \sum_{j=0}^{k-\lceil\nu\rceil} c_{kj} L_j^{(\alpha)}(x), \tag{18}$$

where c_{kj} can be obtained using (10) where $u(x) = x^{k-\nu}$ then,

$$c_{kj} = \frac{\Gamma(j+1)}{\Gamma(j+1+\alpha)} \int_0^\infty x^{k+\alpha-\nu} e^{-x} L_j^{(\alpha)}(x) dx, \quad j = 0, 1, 2, \dots. \tag{19}$$

then,

$$x^{k-\nu} = \sum_{j=0}^{k-\lceil\nu\rceil} \frac{(-1)^j \, (k-\nu)! \, (k-\nu+\alpha+1)!}{(k-\nu-j)! \, (\alpha+j)!} L_j^{(\alpha)}(x). \tag{20}$$

Therefore, the Caputo fractional derivative $D^\nu L_i^{(\alpha)}(x)$ in Eq.(16) can be rewritten in the following form

$$D^\nu L_i^{(\alpha)}(x) = \sum_{k=\lceil\nu\rceil}^{i} \sum_{j=0}^{k-\lceil\nu\rceil} \frac{(-1)^{j+k} \, (\alpha+i)! \, (k-\nu+\alpha)!}{(i-k)! \, (\alpha+k)! \, (k-\nu-j)! \, (\alpha+j)!} L_j^{(\alpha)}(x), \tag{21}$$

for $i = \lceil\nu\rceil, \lceil\nu\rceil+1, \dots, m$. Eq.(21) leads to the desired result. □

Theorem 4. [12]

The error in approximating $D^\nu u(x)$ by $D^\nu u_m(x)$ is bounded by

$$|E_T(m)| \le \sum_{i=m+1}^{\infty} c_i \Pi_\nu(i,j) \, \frac{(\alpha+1)_j}{j!} \, e^{x/2}, \;\; \alpha \ge 0, \;\; x \ge 0, \;\; j = 0, 1, 2, \dots, \tag{22}$$

$$|E_T(m)| \le \sum_{i=m+1}^{\infty} c_i \Pi_\nu(i,j) \left(2 - \frac{(\alpha+1)_j}{j!}\right) e^{x/2}, \;\; -1 < \alpha \le 0, \;\; x \ge 0, \;\; j = 0, 1, 2, \dots. \tag{23}$$

$$\text{where,} \quad \Pi_\nu(i,j) = \sum_{k=\lceil\nu\rceil}^{i} \sum_{j=0}^{k-\lceil\nu\rceil} \Omega_{ijk} \quad \text{and} \quad |E_T(m)| = |D^\nu u(x) - D^\nu u_m(x)|.$$

5. Implementation of Laguerre spectral method for solving FRDE

In this section, we introduce a numerical algorithm using Laguerre spectral method for solving the fractional Riccati differential equation of the form (1).

The procedure of the implementation is given by the following steps:

1. Approximate the function u(t) using the formula (11) and its Caputo fractional derivative $D^\nu u(t)$ using the presented formula (12) with m = 5, then the FRDE (1) is transformed to the following approximated form

$$\sum_{i=1}^{5} \sum_{k=1}^{i} c_i \, w_{i,k}^{(\nu)} \, t^{k-\nu} + \left(\sum_{i=0}^{5} c_i \left(L_i^{(\alpha)}(t) \right) \right)^2 - 1 = 0, \tag{24}$$

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where $w_{i,k}^{(\nu)}$ is defined in (13).

2. The initial condition (2) is given by following form

$$\sum_{i=0}^{5} c_i \left(L_i^{(\alpha)}(0)\right) = u_0. \tag{25}$$

The Eqs.(24)-(25) represent a system of non-linear algebraic equations which contains 6 equations for the unknowns ci, i = 0,1,...,5.

3. Solve the previous system using the Newton iteration method to obtain the unknowns ci, i = 0,1,...,5.

Therefore, the approximate solution will take the same form

$$u(t) = c_0 L_0^{(\alpha)}(t) + c_1 L_1^{(\alpha)}(t) + c_2 L_2^{(\alpha)}(t) + c_3 L_3^{(\alpha)}(t) + c_4 L_4^{(\alpha)}(t) + c_5 L_5^{(\alpha)}(t).$$

The numerical result of the proposed problem (1) is given in figures 1 and 2 with different values of in the interval [0; 3]. Where in figure 1, we presented a comparison between the behavior of the exact solution and the approximate solution using the introduced technique at = 1; u_0 = 0. But, in figure 2 we presented the behavior of the approximate solution with different values of (v = 0:5; 0:75).

The numerical result of the proposed problem (1) is given in figures 1 and 2 with different values of *v* in the interval [0; 3]. Where in figure 1, we presented a comparison between the behavior of the exact solution and the approximate solution using the introduced technique at v= 1; u_0 = 0. But, in gure 2 we presented the behavior of the approximate solution with di erent values of v (v = 0:5; 0:75).

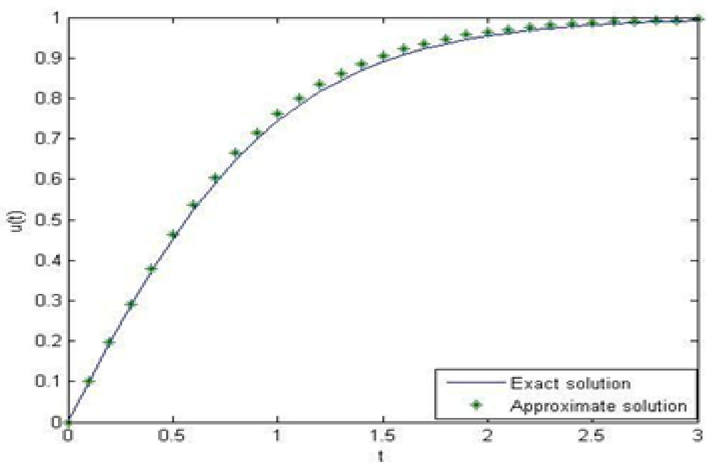

Figure 1: The behavior of the approximate solution using the proposed method and the exact solution at = 1, u0 = 0.

ISSN2305-8269
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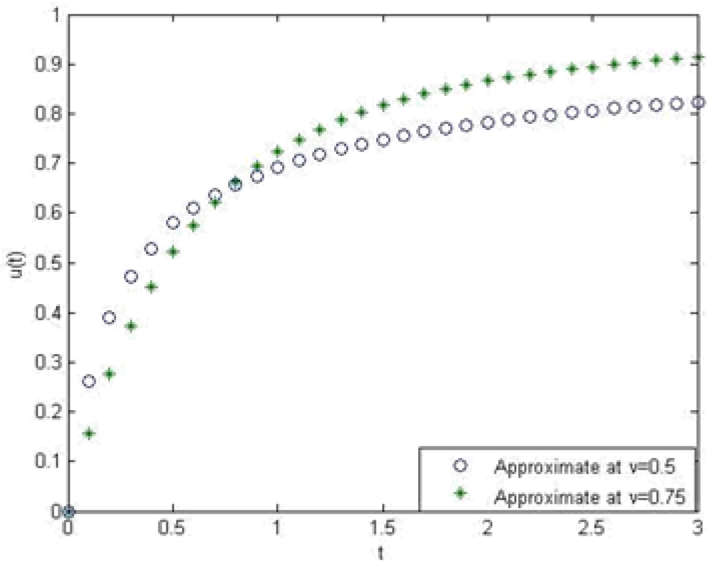

Figure 2: The behavior of the approximate solution with di erent values of .

From these figures we can conclude that the obtained numerical solutions are in excellent agreement with the exact solution.

6. Conclusion

In this article, we introduced an a new spectral collocation method based on Laguerre polyno-mials for solving FRDE. We have introduced an approximate formula for the Caputo fractional derivative of the generalized Laguerre polynomials in terms of classical Laguerre polynomials themselves. In the proposed method we used the properties of the Laguerre polynomials to reduce FRDE to solve a system of algebraic equations. The error upper bound of the proposed approximate formula is stated and proved. The results show that the algorithm converges as the number of m terms is increased. The solution is expressed as a truncated Laguerre series and so it can be easily evaluated for arbitrary values of time using any computer program without any computational e ort. From illustrative examples, we can conclude that this approach can obtain very accurate and satisfactory results. Comparisons are made between approximate solu-tions and exact solutions to illustrate the validity and the great potential of the technique. All computations are done using Matlab 8.

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