Feb, 2014. Vol. 4, No. 8 ISSN2305-8269
International Journal of Engineering and Applied Sciences

www.eaas-journal.org

AVERAGE POSITIVE LYAPUNOV EXPONENT CHARACTERIZATION OF THE PARAMETER PLANE OF HARMONICALLY EXCITED PENDULUM

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ABSTRACT

It is the aim of this paper to further the study of the chaotic behaviour of a harmonically excited pendulum numerically by characterization using the Lyapunov exponent obtained by implementing the Gram-Schmidt orthogonalisation process over series of parameters. A computer program (FORTRAN 95) simulating the physical system supports the numerical investigation. The study also uses the popular fourth order Runge-Kutta scheme with constant integration time step from prescribed initial conditions and fixed drive amplitude, and for large number of drive period. Simulated results are used to create Poincaré sections for selected validation cases to validate FORTRAN coded programmes used for this study, then, chaos diagrams are plotted from various parameters combination.

Keywords: Harmonically excited pendulum, Average Lyapunov exponent, Runge-Kutta, characterization, Grahm Schmidt orthogonalisation, Chaos Diagrams.

INTRODUCTION

Chaos was the law of nature; order was the dream of man (Henry Adams). The irregular and unpredictable time evolution of many nonlinear systems has been dubbed ‘chaos’. It occurs in mechanical oscillators such as vibrating bodies or pendula, in rotating or heated fluid. Its central characteristic is that the system does not repeat its past pattern (even approximately). [3]

Chaos is an aspect of nonlinear dynamics that is concerned with systems governed by equations in which a small change in one variable can induce an unpredictable change in the system.

[1] Argues that Discoveries have been made in laboratory and mathematical models that describe a wide range of variety of systems. This has led to the growing interests in engineering dynamics in modelling systems that are chaotic. [9] Explains that A chaotic system is one in which long-term prediction of the system’s state is impossible due to the fact that the ever present uncertainty in determining its initial state grows exponentially fast in time.

To observe the state variables (time series), the Poincaré map and the average Lyapunov exponents are used to detect chaos in dynamical systems. In this paper, the authors adopted all these tools using a harmonically excited pendulum as a case study.

The rates of orbital divergence or convergence, called Lyapunov exponents are clearly of fundamental importance in studying chaos. They provide a measure of how two trajectories that start from close initial conditions differ as time progresses.

The use of Lyapunov exponent will significantly ease stress, time and cost of experimentation in pendulum analysis as it cautiously and intelligently helps track and filters regions of chaotic and non-chaotic behaviour.

Lyapunov exponent represent the natural way to measure the rapidity of divergence or convergence of a set of infinitesimally close points carried along by a chaotic flow. Positive Lyapunov exponent indicates chaos and divergence of neighbouring trajectories. This divergence is exponential in time. Two initially close orbits in a system with positive Lyapunov exponent will separate very quickly. After separation, the two solutions grow more dissimilar until they are completely different. Negative Lyapunov exponent sets the time scale on which transients or perturbations of the system’s state will decay.

The Lyapunov exponent of a map may be used to obtain a measure of the sensitive dependence upon initial conditions that is characteristic of chaotic behaviour. This exponent (often written as λ) may be readily computed for one-dimensional map such as the

ISSN2305-8269
International Journal of Engineering and Applied Sciences

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logistic map. If the system is allowed to evolve from two slightly differing initial states, say χ and $\chi+\varepsilon$, then after n iterations their divergence may be characterized approximately as;

$\varepsilon(n) \approx \varepsilon e^{\lambda n}$ where the Lyapunov exponent λ gives the average rate of divergence (the average must be taken over many 'initial conditions' spread over the trajectory). If λ is negative, slightly separated trajectories converge and the evolution is not chaotic. If λ is positive, nearby trajectories diverge; the evolution is sensitive to the initial condition and therefore chaotic. Lyapunov exponent are also defined for continuous time dynamical systems such as the pendulum. [3]

To understand the harmonically excited simple pendulum system, it is essential to simulate it numerically. This can be done by using the Runge-Kutta method. [8]

The Runge-Kutta method is a fourth-order integration method. [7] States: basically, the Runge-Kutta method extrapolates each successive point over a discrete time interval. Its simplicity, accuracy, stability at large time steps and relatively quick implementation makes this method a better option. One of the reasons why the Runge-Kutta method (RK4) method is so popular is that the number of function evaluations per step in it equals the order of the method. [4]

The use of Runge-Kutta methods have provided reliable alternative numerical tools for producing fractals, phase plots, bifurcation diagrams and other forms of chaos diagrams.

Poincaré sections are useful when analysing chaotic systems, as they make it easier to understand their dynamics. [6]

Many studies have been carried out on pendulum, its stability, changes from periodic to chaotic behaviour amongst others. With careful and extensive consultation of numerous literatures it shows that there exists a lacuna, and this is that chaos diagrams have not been significantly explored for pendulum. This paper intends to address this research gap.

METHODOLOGY

This study is purely based on numerical simulation of the transformed governing equation for the motion of harmonically excited pendulum given by (1) and (2) simultaneously using the popular fourth order Runge-Kutta scheme with constant integration time step from prescribed initial conditions and fixed drive amplitude, and for large number of drive period. Simulated results are used to create Poincaré sections for selected validation cases, after which chaos diagrams for various parameters combination are plotted.

Harmonically Excited Pendulum

The studied governing equation for the motion of a harmonically excited pendulum is given by equation (1).

$$\frac{d^2\theta}{dt^2} + \frac{1}{q}\frac{d\theta}{dt} + \sin\theta = g\cos(\omega_D t) \qquad (1)$$

The transformation of equation (1) to a pair of first order differential equation produces (2) and (3) under the assumption that ($\theta_1 =$ angular displacement and $\theta_2 =$ angular velocity)

$$\dot{\theta_1} = \theta_2 \qquad (2)$$

$$\dot{\theta_2} = g\cos(\omega_D t) - \frac{1}{q}\theta_2 - \sin(\theta_1) \quad (3)$$

The differential equation contains three adjustable parameters: the driving force amplitude g, the damping or quality parameter q and the angular drive frequency (ω_D).

Fourth order runge-kutta scheme

Using equation (2) and (3) above

$$\dot{\theta_1} = \theta_2 \qquad (2)$$

$$\dot{\theta_2} = g\cos(\omega_D t) - \frac{1}{q}\theta_2 - \sin(\theta_1) \quad (3)$$

$$\frac{d^2\theta_1}{dt^2} = g\left(t, \theta_1, \frac{d\theta_2}{dt}\right);\theta_1(t_0) = y_0\left(\frac{d\theta_1}{dt}\right)$$

$$\frac{d\theta_1}{dt} = \theta_2$$

$$\frac{d\theta_2}{dt} = f(t, \theta_1, \theta_2)$$

$$\theta_1(t_0) = y_0$$

$$\theta_2(t_0) = \dot{y}_0$$

$$k_1 = hf\left(t_n, \theta_{1_n}, \theta_{2_n}\right)$$

$$m_1 = hg\left(t_n, \theta_{1_n}, \theta_{2_n}\right)$$

$$k_2 = \left(t_n + \frac{h}{2}, \theta_{1_n} + \frac{k_1}{2}, \theta_{2_n} + \frac{m_1}{2}\right)$$

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$$m_2 = hg\left(t_n + \frac{h}{2}, \theta_{1_n} + \frac{k_1}{2}, \theta_{2_n} + \frac{m_1}{2}\right)$$

$$k_3 = hf\left(t_n + \frac{h}{2}, \theta_{1_n} + \frac{k_2}{2}, \theta_{2_n} + \frac{m_2}{2}\right)$$

$$m_3 = hg\left(t_n + \frac{h}{2}, \theta_{1_n} + \frac{k_2}{2}, \theta_{2_n} + \frac{m_2}{2}\right)$$

$$k_4 = hf\left(t_n + h, \theta_{1_n} + k_3, \theta_{2_n} + m_3\right)$$

$$m_4 = hg\left(t_n + h, \theta_{1_n} + k_3, \theta_{2_n} + m_3\right)$$

$$t_{n+1} = t_n + h$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$z_{n+1} = z_n + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

Description of calculation of average positive Lyapunov by Grahm Schmidt orthogonalisation method.

The Gram-Schmidt procedure is a particular orthogonalisation algorithm. The basic idea is to first orthogonalise each vector with respect to the previous ones; then normalize result to have norm one.

The description of calculation of average positive Lyapunov by Grahm Schmidt orthogonalisation method is fully spelt out in [5]

Time step selection

A constant time step size to seek solutions of ordinary differential equations of this dynamical system was adopted.

Parameter Plane

According to literature research [3] interest focus on the parameter plane defined by

$2.0 \le q \le 4.0$ and $0.9 \le g \le 1.5$

Validation cases

The under-listed parameters were used to test run propose FORTRAN source codes for this study. The FORTRAN-95 coded algorithms were validated using literature written by [2]. The Poincaré section obtained for test cases must agree visually and correspondingly with figures (1) and (2).

Output files

Finally an output file will be created where the various results and filtration processes has been compiled. The series of compilations includes;

- The positive Lyapunov indicating chaos
- The negative Lyapunov indicating no-chaos
- The average positive Lyapunov from coding of Gram Schmidt orthogonal rules
- The Poincaré sections.

Test Case-I

$(q, g, \omega_D) \equiv (2, 1.5, \frac{2}{3})$, Initial conditions (0, 0), Transient and steady solutions period (10, 5000), number of simulation steps within a period (500)

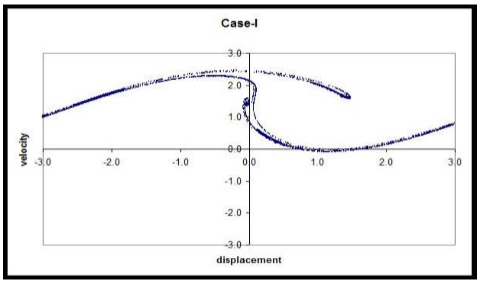

Figure 1: Test case-I

Test Case-II

$(q, g, \omega_D) \equiv (4, 1.5, \frac{2}{3})$, Initial conditions (0, 0), Transient and steady solutions period (10, 5000), number of simulation steps within a period (500)

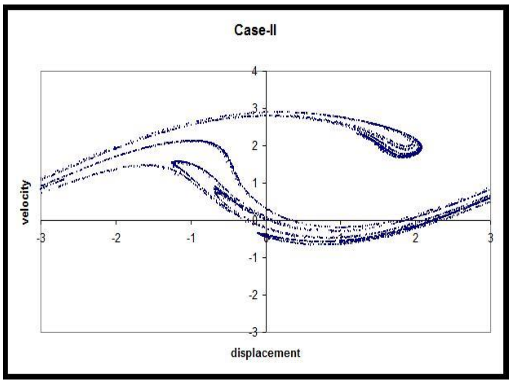

Figure 2: Test case-II

Feb, 2014. Vol. 4, No. 8
ISSN2305-8269
International Journal of Engineering and Applied Sciences

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RESULT AND DISCUSSION

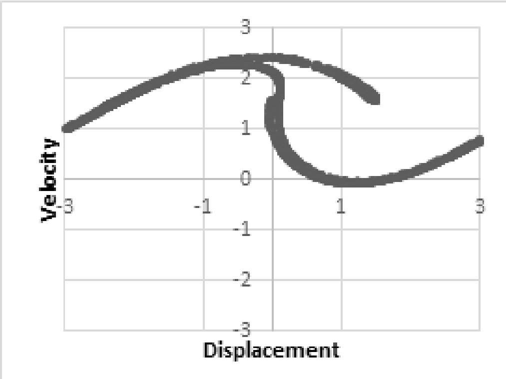

*Figure 3: Test case-I (*Poincaré section*)*

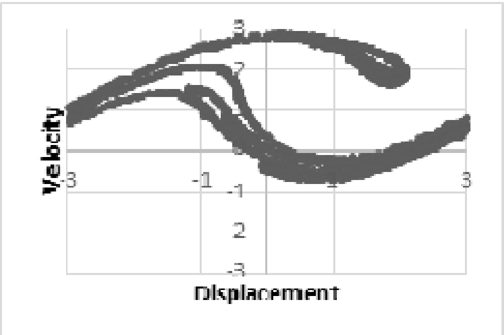

*Figure 4: Test case-II (*Poincaré section*)*

Figure (3) and (4) are Poincaré sections obtained. This is used to validate the coded algorithm and it agrees visually and corresponds to figure (1) and (2).

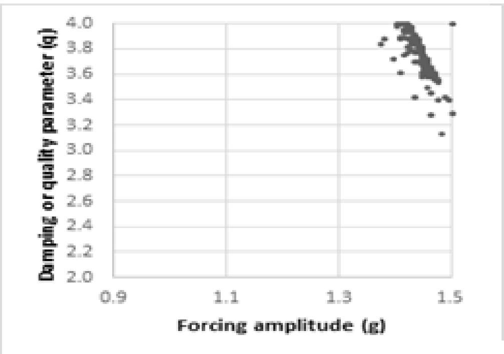

Figure 5: Chaos diagram at 100 by 100 resolution and drive frequency (ω_D=1.0)

In the figure above the probability of parameter combinations that will drive the pendulum chaotically is 0.0087

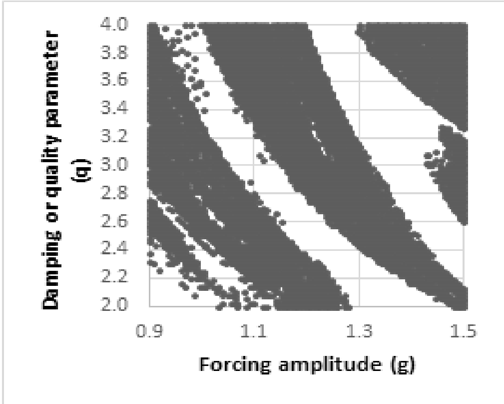

Figure 6: Chaos diagram at 100 by 100 resolution and drive frequency (ω_D=2/3)

In figure (6) the probability of parameter combinations that will drive the pendulum chaotically is 0.54

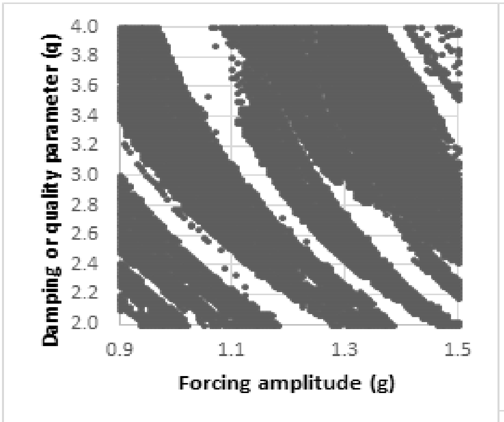

Figure 7: Chaos diagram at 100 by 100 resolution and drive frequency (ω_D=0.6)

In the figure (7) above the probability of parameter combinations that will drive the pendulum chaotically is 0.7

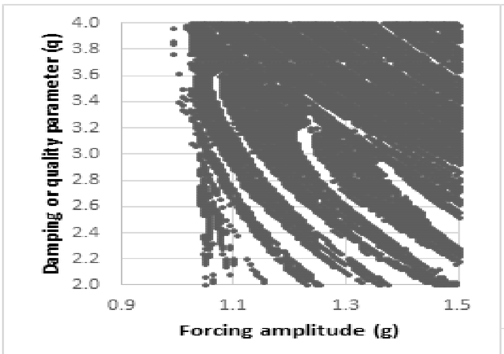

Figure 8: Chaos diagram at 100 by 100 resolution and drive frequency (ω_D =0.1)

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The probability of parameter combinations that will drive the pendulum chaotically is 0.53 in the figure (8) above.

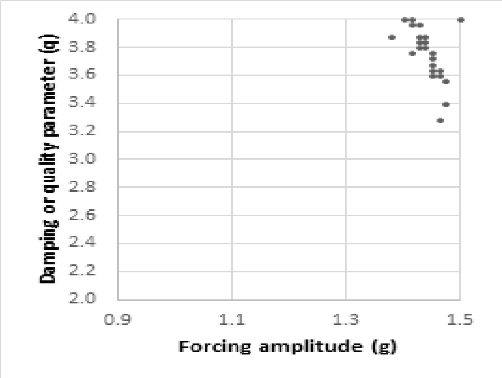

Figure 9: Chaos diagram at 50 by 50 resolution and drive frequency (ω_D=1.0)

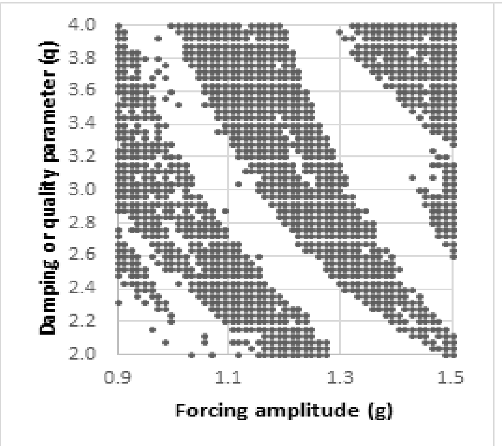

Figure 10: Chaos diagram at 50 by 50 resolution and drive frequency (ω_D=2/3)

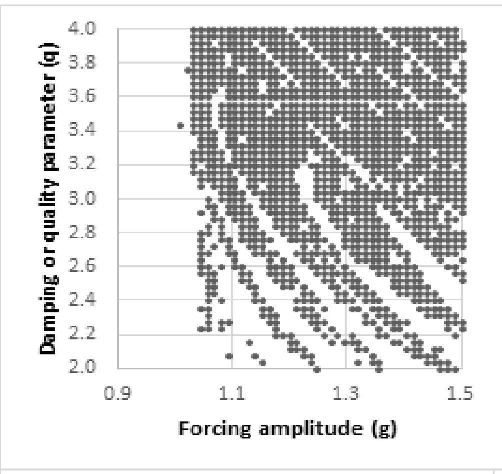

Figure 11: Chaos diagram at 50 by 50 resolution and drive frequency (ω_D=0.1)

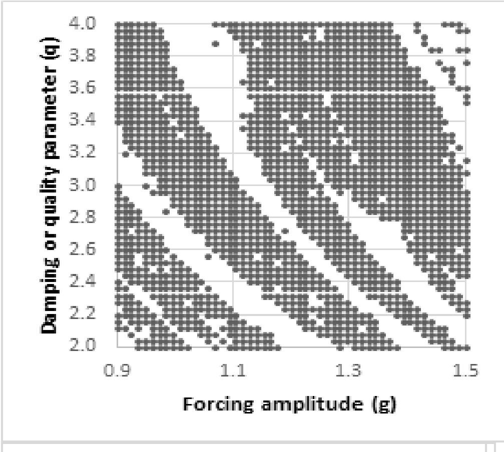

Figure 12: Chaos diagram at 50 by 50 resolution and drive frequency (ω_D=0.6)

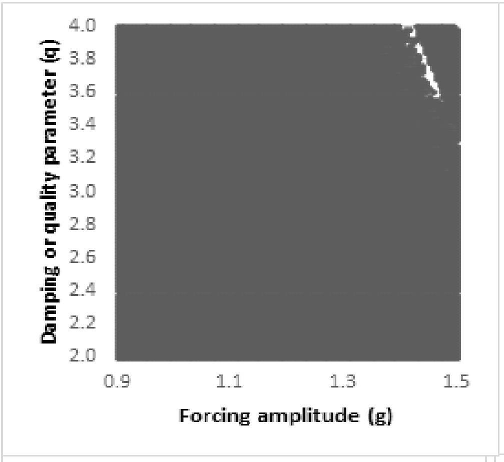

Figure 13: Non-chaos diagram at 100 by 100 resolution and drive frequency (ω_D =1.0)

In the figure above the probability of parameter combinations that will drive the pendulum non-chaotically is 0.99

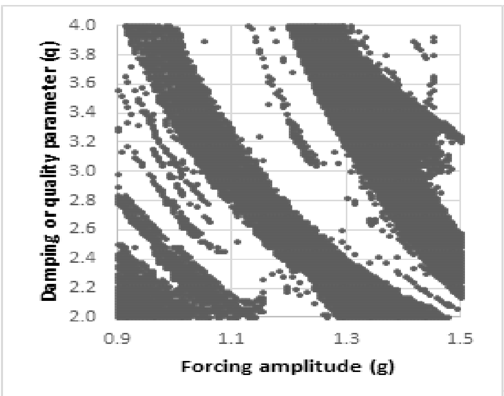

Figure 14: Non-chaos diagram at 100 by 100 resolution and drive frequency (ω_D=2/3)

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In the figure above the probability of parameter combinations that will drive the pendulum non-chaotically is 0.46

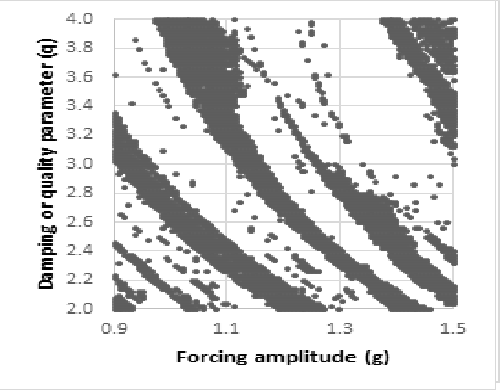

Figure 15: Non-chaos diagram at 100 by 100 resolution and drive frequency (ω_D =0.6)

The probability of parameter combinations that will drive the pendulum non-chaotically is 0.3

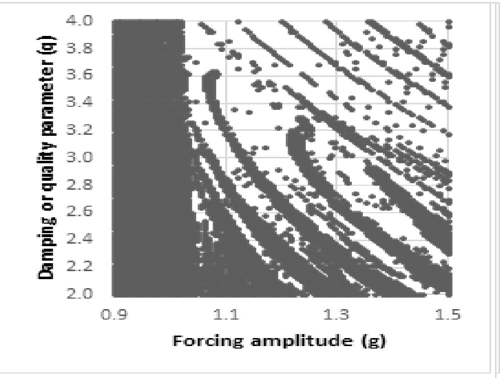

Figure 16: Non-chaos diagram at 100 by 100 resolution and drive frequency (ω_D =0.1)

In the figure above the probability of parameter combinations that will drive the pendulum chaotically is 0.0087

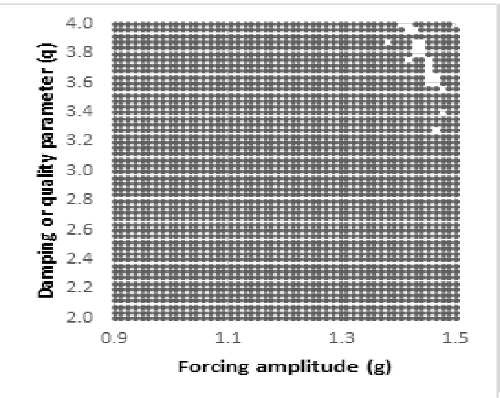

Figure 17: Non-chaos diagram at 50 by 50 resolution and drive frequency (ω_D=1.0)

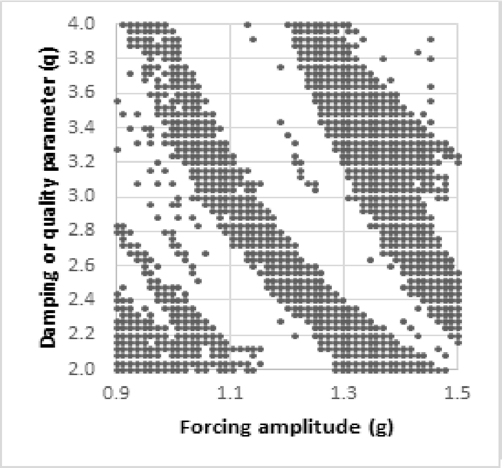

Figure 18: Non-chaos diagram at 50 by 50 resolution and drive frequency (ω_D=2/3)

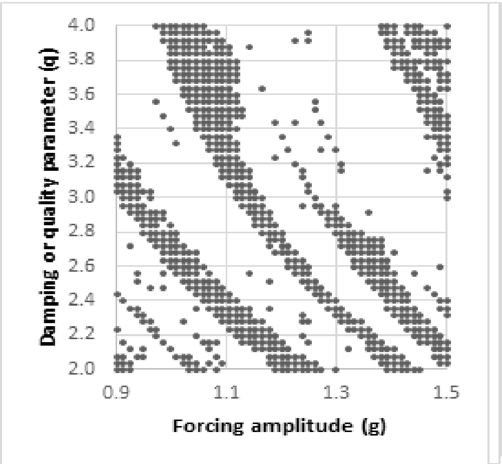

Figure19: Non-chaos diagram at 50 by 50 resolution and drive frequency (ω_D=0.6)

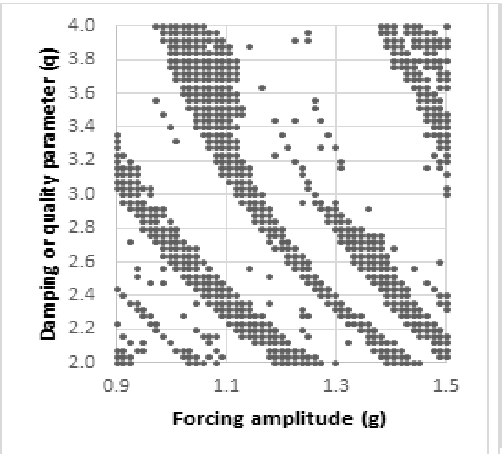

Figure 20: Non-chaos diagram at 50 by 50 resolution and drive frequency (ω_D=0.1)

ISSN2305-8269
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RESULTS DISCUSSION

The Poincare sections obtained for test cases agreed visually and correspondingly satisfactory with literature results. The probability that the parameter combination in the chosen parameter plane drives the pendulum chaotically at a drive frequency of 0.1-0.6 and 2/3 are higher than parameter combination that will drive it periodically/non-chaotically and the probability is highest at 0.6. At drive frequency of 0.7-1.0 and above the probability that parameter combination drives the pendulum non-chaotically/periodically is higher.

CONCLUSION

This study has developed a faster and reliable numerical tool that can predict the chaos diagrams of a harmonically excited pendulum in the damping quality versus forcing amplitudes plane based on the popular fourth order Runge-kutta and the average Lyapunov exponent using Grahm Schmidt orthogonalisation. The study demonstrated the significant utility of numerical techniques particularly in solving nonlinear problems and shows that in addition to being sensitive to initial conditions, chaos is equally sensitivity to appropriate simulation time steps.

Increasing the resolution across the frequency required a longer processing time for the computer but it led to a well-refined and detailed information about parameter combinations that will lead to chaotic or non-chaotic behaviour of the pendulum.

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