EXACT TRAVELING WAVE SOLUTIONS FOR NONLINEAR FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS USING THE IMPROVED (G'/G) – EXPANSION METHOD

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ABSTRACT

In this article, we apply the improved (G'/G)-expansion method to construct the exact solutions of the nonlinear fractional partial differential equations (PDEs) in the sense of the modified Riemann-Liouville derivative. Based on a nonlinear fractional complex transformation, certain fractional partial differential equations can be turned into other nonlinear ordinary differential equations (ODEs) of integer orders. For illustrating the validity of this method, we apply it to four nonlinear fractional PDEs equations namely, the space-time fractional Potential Kadomtsev-Petviashvili (PKP) equation, the space-time fractional Symmetric Regularized Long wave (SRLW) equation, the space-time fractional Sharma-Tasso-Olver (STO) equation and the space-time fractional Kolmogorov-Petrovskii-Piskunov (KPP) equation. This method can be applied to many other nonlinear fractional PDEs in mathematical physics.

Keywords: Nonlinear space-time fractional PDES, Improved (G' / G) - expansion method, Nonlinear fractional complex transformation, Exact solutions; Modified Riemann- Liouville derivative.

INTRODUCTION

In recent years, nonlinear fractional differential equations have been attracted great interest. It is caused by both the development of the theory of fractional calculus itself and by the applications of such constructions in various sciences such as physics, engineering and biology [11,15,20,22,25,27,28]. For better understanding the mechanisms of the complicated nonlinear physical phenomena as well as further applying them in practical life, the solutions of these equation [2,16,19,26,30,31] are much involved. In the past, many analytical and numerical method have been proposed to obtain the solutions of nonlinear fractional differential equations, such as the finite difference method [17], the finite element method [7], the differential transform method [3,21], the Adomian decomposition method [4,5,12,23], the variational iteration method [13,24,29], the homotopy perturbation method [8], the improved (G'/G)-expansion method [6,9,32], the fractional sub-equation method [1,10,33,34] and so on.

In this article, we will apply the improved (G'/G)-expansion method [26-28] for solving the nonlinear fractional PDEs in the sense of the modified Riemann-Liouville derivative obtained in [14,18]. The modified Riemann-Liouville derivative of order $\alpha$ is defined by the following expression:

$$D^{\alpha}_{t} f(t) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{\partial}{\partial t} \right) \left( t - \eta \right)^{\alpha-1} f(\eta) \bigg|_{\eta=0}, \quad 0 < \alpha \leq 1,$$

(1)

$$\left[ f^{(n)}(t) \right]_{t=0}^{t=\gamma}, \quad n \leq \alpha < n + 1, \quad n \geq 1$$

We list some important properties for the modified Riemann-Liouville derivative as follows:
\[ D_i^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}, \ r > 0 \quad (2) \]

\[ D_i^\alpha [f(t)g(t)] = f(t)D_i^\alpha g(t) + g(t)D_i^\alpha f(t) \quad (3) \]

\[ D_i^\alpha [f(g(t))] = f(g(t))D_i^\alpha g(t) = D_x^\alpha f(g(t))[g'(t)]^\alpha \quad (4) \]

The rest of this article is organized as follows: In Sec. 2, the improved \((G'/G)\)-expansion method for solving nonlinear fractional PDEs is given. In Sec. 3, we apply this method to establish the exact solutions for the space-time nonlinear fractional PKP equation, the space-time nonlinear fractional SRLW equation, the space-time nonlinear fractional STO equation and the space-time nonlinear fractional KPP equation. In Sec. 4, some conclusions and discussions are obtained.

**DESCRIPTION OF THE IMPROVED \((G'/G)\)-EXPANSION METHOD FOR SOLVING PDES**

Suppose that we have the following nonlinear fractional partial differential equation:

\[ F(u, D_i^\alpha u, D_x^\alpha u, ...) = 0, \quad 0 < \alpha \leq 1, \quad (5) \]

where \( D_i^\alpha u, D_x^\alpha u \) are the modified Riemann-Liouville derivatives, and \( F \) is a polynomial in \( u(x, t) \) and its partial fractional derivatives, in which the highest order derivatives and the nonlinear terms are involved. In the following, we give the main steps of this method:

**Step1.** Using the nonlinear fractional complex transformation \([33,34]\)

\[ u(x,t) = U(\xi), \quad \xi = \frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{ct^\alpha}{\Gamma(1+\alpha)} + \xi_0 \quad (6) \]

where \( k, c, \xi_0 \) are constants with \( k, c \neq 0 \), to reduce Eq. (5) to the following ordinary differential equation (ODE) with integer order:

\[ P(U, U', U'', ...) = 0, \quad (7) \]

where \( P \) is a polynomial in \( U(\xi) \) and its total derivatives, while the dashes denote the derivatives with respect to \( \xi \).

**Step2.** We assume that Eq. (7) has the formal solution:

\[ U(\xi) = \sum_{i=-m}^{m} a_i \left( \frac{G'(\xi)}{G(\xi)} \right)^i, \quad (8) \]

where \( a_i \) are constants.

**Step3.** The positive integer \( m \) in Eq. (8) can be determined by balancing the highest-order derivatives with the nonlinear terms appearing in Eq. (7).

**Step4.** We substitute (8) along with Eq. (9) into Eq. (7) to obtain polynomials in \( \left( \frac{G'}{G} \right)^i, \ i = 0, \pm 1, \pm 2, ... \). Equating all the coefficients of these polynomials to zero, yields a set of algebraic equations.

**Step5.** We solve the algebraic equations of step 4, using the Maple or Mathematica to find the values of \( a_i, k, c, \lambda, \mu \). Substituting these values into (8) and using the ratios:
where $\phi_1 = \sqrt{\lambda^2 - 4\mu}, \lambda^2 - 4\mu > 0$ and $\phi_2 = \sqrt{4\mu - \lambda^2}, 4\mu - \lambda^2 > 0$. We obtain the exact solutions of Eq. (5), where $c_1$ and $c_2$ are arbitrary constants.

**APPLICATIONS**

In this section we construct the exact solutions of the following four nonlinear fractional PDEs using the proposed method of Sec. 2 as follows:

**Example 1. The space-time nonlinear fractional PKP equation**

This equation is well-known [1] and has the form:

$$\frac{1}{4} D^{4\alpha} u + \frac{3}{2} D^2 u D^{2\alpha} u + \frac{3}{4} D^{2\alpha} u + D^{2\alpha} (D^\alpha u) = 0, \quad (11)$$

where $0 < \alpha \leq 1$. Eq. (11) has been investigated in [1] using the fractional sub-equation method. Let us now solve Eq. (11) using the proposed method of Sec. 2. To this end, we use the nonlinear fractional complex transformation

$$u(x, y, t) = U(\xi),$$

$$\xi = \frac{k_1 x^\alpha}{\Gamma(1 + \alpha)} + \frac{k_2 y^\alpha}{\Gamma(1 + \alpha)} + \frac{ct^\alpha}{\Gamma(1 + \alpha)} + \xi_0, \quad (12)$$

where $k_1, k_2, c, \xi_0$ are constants, to reduce Eq. (11) to the following ODE with integer order:

$$k_1^4 U'''' + 3k_1^3 U''' + (3k_1^2 + 4ck_1) U' = 0. \quad (13)$$

By balancing $U'''$ with $U''^2$, we have $m=1$. Consequently, Eq. (13) has the formal solutions:
\[
\left(\frac{G'}{G}\right) = k'_i(7a_i\mu \lambda^2 + 8a_i\mu^2) + 3k'_i(\lambda^2 - 2a_i\mu + 2a_i\mu^2) + a_i\mu(3k'_i + 4k\mu) = 0,
\]
\[
\left(\frac{G''}{G}\right) = 2a_i\mu^2k'_i + 2a_i\mu k'_i = 0.
\]

On solving these algebraic equations with the aid of Maple or Mathematica we have the following cases:

**Case 1.**
\(\lambda = 0, \quad \mu = \mu, \quad k_1 = k_1, \quad k_2 = k_2,\)
\(c = \frac{1}{4k_i}(16k_i^4\mu - 2k_i^2), \quad a_i = -2k_i\mu, \quad a_i = 2k_i.\)

**Case 2.**
\(\lambda = 0, \quad \mu = \mu, \quad k_1 = k_1, \quad k_2 = k_2,\)
\(c = \frac{1}{4k_i}(4k_i^4\mu - 2k_i^2), \quad a_i = 2k_i, \quad a_i = 0.\)

**Case 3.**
\(\lambda = 0, \quad \mu = \mu, \quad k_1 = k_1, \quad k_2 = k_2,\)
\(c = \frac{-1}{4k_i}(k_i^4(\lambda^2 - 4\mu) + 2k_i^2), \quad a_i = -2k_i\mu, \quad a_i = 0.\)

Let us now write down the following exact solutions of the space-time fractional PKP equation (13) for case 1 (Similarly for cases 2,3 which are omitted here for simplicity):

(i) If \(\mu < 0\) (Hyperbolic function solutions)

In this case, we have the exact solution:

\[
u(x, y, t) = 2k_i\sqrt{-\mu}\left\{c_i\sinh(\sqrt{-\mu}t) + c_i\cosh(\sqrt{-\mu}t)\right\}\]

(ii) If \(\mu > 0\) (Trigonometric function solutions)

In this case we have the exact solution:

\[
u(x, y, t) = 2k_i\sqrt{\mu}\cos(\sqrt{\mu}t) + a_i\sinh(\sqrt{-\mu}t) + a_i\cosh(\sqrt{-\mu}t)\]

while if we set \(c_2 = 0\) and \(c_1 \neq 0\), in (15) we have the solitary wave solution:

\[
u_2(x, y, t) = u_1(x, y, t),\]

(iii) If \(\mu = 0\) (Rational function solutions)

In this case, we have the exact solution:

\[
u(x, y, t) = 2k_i\sqrt{-\mu}\left\{c_i\sin(\sqrt{-\mu}t) + c_i\cos(\sqrt{-\mu}t)\right\}\]

If we set \(c_1 = 0\) and \(c_2 \neq 0\) in (15) we have the solitary solution:

\[
u_1(x, y, t) = 2k_i\sqrt{\mu}\coth(\sqrt{\mu}t) + a_i + 2k_i\sqrt{-\mu}\tanh(\sqrt{-\mu}t),\]

while if we set \(c_2 = 0\) and \(c_1 \neq 0\), in (15) we have the solitary wave solution:

\[
u_3(x, y, t) = u_3(x, y, t),\]

where

\[
\xi = k_i^x + \frac{k_i^y}{\Gamma(1 + \alpha)} + \frac{1}{4k_i^2} (16k_i^4\mu - 2k_i^2) t + \frac{\tau}{\Gamma(1 + \alpha)} + \xi \]

(iii) If \(\mu = 0\) (Rational function solutions)

In this case, we have the exact solution:

\[
u(x, y, t) = 2k_i\left\{c_i\left(\frac{c}{c_i + c_i}\right) + a_i - 2\mu k_i\left(\frac{c}{c_i + c_i}\right)\right\}\]
where
\[
\xi = \frac{k_1 x^\alpha}{\Gamma(1+\alpha)} + \frac{k_2 y^\alpha}{\Gamma(1+\alpha)} - \frac{k_2^2}{2} \frac{t^\alpha}{\Gamma(1+\alpha)} + \xi_0
\]

Example 2. The space-time nonlinear fractional SRLW equation

This equation is well-known [1] and has the form:

\[
D_t^{\alpha} u + D_x^{\alpha} u + u D_t^{\alpha} (D_x^{\alpha} u) + D_t^{\alpha} u D_x^{\alpha} u + D_x^{\alpha} (D_t^{\alpha} u) = 0, \tag{22}
\]

where \(0 < \alpha \leq 1\). Eq. (22) has been investigated in [1] using the fractional sub-equation method. Let us solve Eq. (22) using the proposed method of Sec. 2. To this end, we use the nonlinear fractional complex transformation

\[
u(x,t) = U(\xi), \quad \xi = \frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{ct^\alpha}{\Gamma(1+\alpha)} + \xi_0, \tag{23}
\]

where \(k, c, \xi_0\) are constants, to reduce Eq. (22) to the following ODE with integer order:

\[
k^2 c^2 U^{\prime\prime} + (k^2 + c^2) U + \frac{k c}{2} U^2 = 0, \tag{24}
\]

By balancing \(U^{\prime\prime}\) with \(U^2\), we have \(m=2\). Consequently, Eq. (24) has the formal solutions:

\[
U(\xi) = a_2 \left( \frac{G'}{G} \right)^i + a_1 \left( \frac{G'}{G} \right)^i + a_0 + a_1 \left( \frac{G'}{G} \right)^i + a_1 \left( \frac{G'}{G} \right)^i, \tag{25}
\]

where \(a_2, a_1, a_0, a_{-1}, a_{-2}\) are constants to be determined later, such that \(a_{-2} \neq 0\) or \(a_{-2} = 0\). Substituting Eq. (25) along with Eq. (9) into Eq. (24), collecting all the terms of the same orders \(\left( \frac{G'}{G} \right)^i, i = 0, \pm 1, \pm 2, \ldots \) and setting each coefficient to zero, we have the following set of algebraic equations:

On solving the above set of algebraic equations with the aid of Maple or Mathematica we have the following cases:

Case 1.
λ = λ,  c = c,  k = k,  μ = \frac{1}{4k^2c^2}(\lambda^2k^2c^2 - (c^2 + k^2)),

a_1 = \frac{-3\lambda}{kc}[\lambda^2k^2c^2 - (c^2 + k^2)],  a_0 = a_3 = 0,

a_2 = \frac{-3}{4k^2c^2}(\lambda^2k^2c^2 - (c^2 + k^2))^\prime,  a_0 = a_3 = a_i = -12kc

Case 2.

c = c,  k = k,  μ = \frac{-1}{576kc^2}[144(c^2 + k^2) - a^2_0],  \lambda = \frac{-a_0}{12kc},

a_0 = \frac{1}{48kc}[48(c^2 + k^2) - a^2_0],  a_0 = a_2 = a_1 = a_3 = -12kc

Let us now write down the following exact solutions of the space-time fractional PKP equation (22) for case 1 (Similarly for case 2 which is omitted here for simplicity):

(i) If \( \lambda^2 - 4\mu > 0 \) (Hyperbolic function solutions)

In this case, we have the exact solution:

\[
 u(x,t) = \left[ \frac{3\lambda^2k^2c^2 - (k^2 + c^2)}{kc} \right]^{1/2} \left[ \begin{array}{c}
 \frac{-3\lambda}{2k^2c^2} + \frac{\sqrt{k^2 + c^2}}{2k^2c^2} + \frac{c}{k^2c^2} \sinh \left( \frac{\sqrt{k^2 + c^2}}{2k^2c^2} \right) \\
 \frac{-3\lambda}{2k^2c^2} - \frac{\sqrt{k^2 + c^2}}{2k^2c^2} + \frac{c}{k^2c^2} \cosh \left( \frac{\sqrt{k^2 + c^2}}{2k^2c^2} \right)
\end{array} \right]^{2/3}.
\]

(26)

If we set \( c_1 = 0 \) and \( c_2 \neq 0 \) in (26) we have the solitary wave solution:

\[
 u(x,t) = \left[ \frac{3\lambda^2k^2c^2 - (k^2 + c^2)}{kc} \right]^{1/2} \left( \begin{array}{c}
 \frac{-3\lambda}{2k^2c^2} + \frac{\sqrt{k^2 + c^2}}{2k^2c^2} + \frac{c}{k^2c^2} \sinh \left( \frac{\sqrt{k^2 + c^2}}{2k^2c^2} \right) \\
 \frac{-3\lambda}{2k^2c^2} - \frac{\sqrt{k^2 + c^2}}{2k^2c^2} + \frac{c}{k^2c^2} \cosh \left( \frac{\sqrt{k^2 + c^2}}{2k^2c^2} \right)
\end{array} \right]^{2/3}.
\]

(27)

while if we set \( c_2 = 0 \) and \( c_1 \neq 0 \), in (26) we have the solitary wave solution:

\[
 u(x,t) = \left[ \frac{3\lambda^2k^2c^2 - (k^2 + c^2)}{kc} \right]^{1/2} \left( \begin{array}{c}
 \frac{-3\lambda}{2k^2c^2} + \frac{\sqrt{k^2 + c^2}}{2k^2c^2} + \frac{c}{k^2c^2} \sinh \left( \frac{\sqrt{k^2 + c^2}}{2k^2c^2} \right) \\
 \frac{-3\lambda}{2k^2c^2} - \frac{\sqrt{k^2 + c^2}}{2k^2c^2} + \frac{c}{k^2c^2} \cosh \left( \frac{\sqrt{k^2 + c^2}}{2k^2c^2} \right)
\end{array} \right]^{2/3}.
\]

(28)

If \( c_2 \neq 0 \) and \( c_1 < c_2^2 \), then we have the solitary wave solution:

\[
 u(x,t) = \left[ \frac{3\lambda^2k^2c^2 - (k^2 + c^2)}{kc} \right]^{1/2} \left( \begin{array}{c}
 \frac{-3\lambda}{2k^2c^2} + \frac{\sqrt{k^2 + c^2}}{2k^2c^2} + \frac{c}{k^2c^2} \sinh \left( \frac{\sqrt{k^2 + c^2}}{2k^2c^2} \right) \\
 \frac{-3\lambda}{2k^2c^2} - \frac{\sqrt{k^2 + c^2}}{2k^2c^2} + \frac{c}{k^2c^2} \cosh \left( \frac{\sqrt{k^2 + c^2}}{2k^2c^2} \right)
\end{array} \right]^{2/3}.
\]

(29)

where \( \xi_j = \sinh^{-1} \left( \frac{c}{c_j} \right) \), while if \( c_1 \neq 0 \) and

\( c_2^2 < c_1^2 \), then we have the solitary wave solution:

\[
 u(x,t) = \left[ \frac{3\lambda^2k^2c^2 - (k^2 + c^2)}{kc} \right]^{1/2} \left( \begin{array}{c}
 \frac{-3\lambda}{2k^2c^2} + \frac{\sqrt{k^2 + c^2}}{2k^2c^2} + \frac{c}{k^2c^2} \sinh \left( \frac{\sqrt{k^2 + c^2}}{2k^2c^2} \right) \\
 \frac{-3\lambda}{2k^2c^2} - \frac{\sqrt{k^2 + c^2}}{2k^2c^2} + \frac{c}{k^2c^2} \cosh \left( \frac{\sqrt{k^2 + c^2}}{2k^2c^2} \right)
\end{array} \right]^{2/3}.
\]

(30)
where \( \xi_1 = \coth^{-1} \left( \frac{c_2}{c_1} \right) \) and

\[
\xi = \frac{k x^\alpha}{\Gamma(1 + \alpha)} + \frac{c t^\alpha}{\Gamma(1 + \alpha)} + \xi_0.
\]

**Exampe3. The space-time nonlinear fractional STO equation**

This equation is well-known [34] and has the form:

\[
D^\alpha u + 3\beta(D^\alpha u)^3 + 3\beta u D^\alpha u + 3\beta u D^\alpha u + \beta D^\alpha u = 0, \tag{31}
\]

where \( 0 < \alpha \leq 1 \). Eq. (31) has been investigated in [34] using the fractional sub-equation method. Let us now solve Eq. (31) using the proposed method of Sec. 2. To this end, we use the transformation (6) to reduce Eq. (31) to the following ODE with integer order:

\[
c U + 3\beta k^3 U u' + \beta k U^3 + \beta k^3 U^* = 0, \tag{32}
\]

By balancing \( U^* \) with \( U^3 \), we have \( m=1 \). Consequently, Eq. (31) has the formal solutions:

\[
U(\xi) = a_1 \left( \frac{G'}{G} \right) + a_0 + a_{-1} \left( \frac{G'}{G} \right)^{-1}, \tag{33}
\]

where \( a_1, a_0, a_{-1} \) are constants to be determined later, such that \( a_{-1} \neq 0 \) or \( a_1 \neq 0 \). Substituting (33) along with Eq. (9) into (32), collecting all the terms of the same order \( \left( \frac{G'}{G} \right)^i, i = 0, \pm 1, \pm 2, \pm 3 \) and setting each coefficient to zero, we have the following set algebraic equations:

\[
\left( \frac{G'}{G} \right)^i : -3\beta a_1 k^3 + \beta a_1 k + 2\beta a_1 k = 0,
\]

\[
\left( \frac{G'}{G} \right)^i : -3\beta k^3 (a_1 + a_0 a_1) + 3\beta a_1 a_1 k + 3\beta a_1 \lambda k = 0,
\]

\[
\left( \frac{G'}{G} \right)^i : ca_1 - 3\beta k^3 (a_1 + a_1 a_1) + 3\beta k (a_1 a_1 + a_1 a_1) + \beta k^3 (2a_1 + a_1 a_1) = 0.
\]

By solving these algebraic equations with the aid of Maple or Mathematica we have the following cases:

**Case 1.**

\[
\lambda = \lambda, \quad \mu = \mu, \quad \beta = \beta, k = k, c = -\beta k^3 (\lambda^2 - 4\mu),
\]

\[
a_i = 2k, \quad a_0 = \lambda k, \quad a_{-1} = 0
\]

**Case 2.**

\[
\lambda = \lambda, \quad \mu = \mu, \quad \beta = \beta, k = k, c = -\beta k^3 (\lambda^2 - 4\mu),
\]

\[
a_i = 0, \quad a_{-1} = -2\mu k, \quad a_0 = -k \lambda
\]

**Case 3.**

\[
\lambda = \lambda, \quad \mu = \mu, \quad \beta = \beta, k = k, c = -\beta k^3 (\lambda^2 - 4\mu),
\]

\[
a_i = 0, \quad a_{-1} = -2\mu k, \quad a_0 = -k \lambda
\]

Let us now write down the following exact solutions of the space-time fractional STO equation (31) for
case 1 (Similarly for cases 2, 3 which are omitted here for simplicity):

(i) If \( \lambda^2 - 4\mu > 0 \) (Hyperbolic function solutions)

In this case, we have the exact solution:

\[
\begin{aligned}
  u(x, t) &= k \sqrt{\lambda^2 - 4\mu} \left\{ c \cos \left( \frac{\sqrt{\lambda^2 - 4\mu} x}{2} \right) + c \sin \left( \frac{\sqrt{\lambda^2 - 4\mu} x}{2} \right) \right\}, \\
  &\quad \text{if} \ c_1 = 0 \quad \text{and} \quad c_2 \neq 0, \quad \text{in (34)} \\
\end{aligned}
\]

If we set \( c_1 = 0 \) and \( c_2 \neq 0 \) in (34) we have the solitary solution:

\[
\begin{aligned}
  u_1(x, t) &= k \sqrt{\lambda^2 - 4\mu} \tanh \left( \frac{\sqrt{\lambda^2 - 4\mu} x}{2} \xi \right), \\
  \text{while if we set} \ c_2 = 0 \quad \text{and} \quad c_1 \neq 0, \quad \text{in (34)} \quad \text{we have the solitary wave solution:} \\
  u_2(x, t) &= k \sqrt{\lambda^2 - 4\mu} \coth \left( \frac{\sqrt{\lambda^2 - 4\mu} x}{2} \xi \right). \\
\end{aligned}
\]

If \( c_2 \neq 0 \) and \( c_1^2 < c_2^2 \), then we have the solitary wave solution:

\[
\begin{aligned}
  u_3(x,t) &= k \sqrt{\lambda^2 - 4\mu} \coth \left( \frac{\sqrt{\lambda^2 - 4\mu} x}{2} \xi \xi \right), \\
  \text{where} \quad \xi_1 &= \tanh^{-1} \left( \frac{c_1}{c_2} \right), \quad \text{while if} \ c_1 \neq 0 \quad \text{and} \\
  c_2^2 < c_1^2, \quad \text{then we have the solitary wave solution:} \\
  u_4(x,t) &= k \sqrt{\lambda^2 - 4\mu} \tanh \left( \xi_1 + \frac{\xi_1}{2} \sqrt{\lambda^2 - 4\mu} \right), \\
  \text{where} \quad \xi_1 &= \cosh^{-1} \left( \frac{c_2}{c_1} \right). \\
\end{aligned}
\]

(ii) If \( \lambda^2 - 4\mu < 0 \) (Trigonometric function solutions)

In this case we have the exact solution:

\[
\begin{aligned}
  u(x, t) &= k \sqrt{4\mu - \lambda^2} \left\{ -c \sin \left( \frac{\sqrt{4\mu - \lambda^2} x}{2} \right) + c \cos \left( \frac{\sqrt{4\mu - \lambda^2} x}{2} \right) \right\}, \\
  \text{if we set} \ c_1 = 0 \quad \text{and} \quad c_2 \neq 0 \quad \text{in (39)} \quad \text{we have the periodic solution:} \\
  u_1(x, t) &= k \sqrt{4\mu - \lambda^2} \cot \left( \frac{\sqrt{4\mu - \lambda^2} x}{2} \right), \\
  \text{while if we set} \ c_2 = 0 \quad \text{and} \quad c_1 \neq 0, \quad \text{in (39)} \quad \text{we have the periodic solution:} \\
  u_2(x, t) &= -k \sqrt{4\mu - \lambda^2} \tan \left( \frac{\sqrt{4\mu - \lambda^2} x}{2} \right), \\
  \text{If} \ c_2 \neq 0 \quad \text{and} \quad c_1^2 < c_2^2, \quad \text{then we have the periodic solution:} \\
  u_3(x, t) &= k \sqrt{4\mu - \lambda^2} \cot \left( \xi_1 + \frac{\xi_1}{2} \sqrt{4\mu - \lambda^2} \right), \\
  \text{where} \quad \xi_1 &= \tan^{-1} \left( \frac{c_1}{c_2} \right), \quad \text{while if} \ c_1 \neq 0 \quad \text{and} \\
  c_2^2 < c_1^2, \quad \text{then we have the periodic solution:} \\
  u_4(x, t) &= k \sqrt{4\mu - \lambda^2} \tan \left( \xi_1 + \frac{\xi_1}{2} \sqrt{4\mu - \lambda^2} \right), \\
  \text{where} \quad \xi_1 &= \cot^{-1} \left( \frac{c_2}{c_1} \right) \quad \text{and} \\
  \xi &= \frac{kx^\alpha}{\Gamma(1 + \alpha) - \beta k^3(\lambda^2 - 4\mu) - i^\alpha}{\Gamma(1 + \alpha) + \xi_0}. \\
\end{aligned}
\]

Example 4

The space-time nonlinear fractional KPP equation

This equation is well-known [6] and has the form:

\[
\begin{aligned}
  D_t^\alpha u - D_x^{2\alpha} u + \mu u + \gamma u^2 + \delta u^3 &= 0, \\
  \end{aligned}
\]

where \( 0 < \alpha \leq 1 \) and \( \mu, \gamma, \delta \) are non zero constants. This equation is important in the physical
fields, and it includes the Fisher equation. Huxlay equation, Burgers equation, Chaffee-Infanmf equation and Fitzhugh-Nagumo equation. When \( \alpha = 1 \), Eq. (44) has been discussed in [6] using the \((G'/G)\) expansion method. Let us solve Eq. (44) using the proposed method of Sec. 2. To this end, we use the nonlinear fractional complex transformation (6), to reduce Eq. (44) to the following ODE with integer order:

\[
c U^\prime - k^2 U'' + \mu U + \gamma U^2 + \delta U^3 = 0, \quad (45)
\]

By balancing \( U'' \) with \( U^3 \), we have \( m=1 \). Consequently, Eq. (45) has the same formal solution (44). Substituting (44) along with Eq. (9) into Eq. (43), collecting all the terms of the same orders \((G')^i/G\), \( i = 0, \pm 1, \pm 2, \pm 3 \) and setting each coefficient to zero, we have the following set of algebraic equations:

\[
\begin{align*}
(G')^0/G & : \quad -2a_k k^2 + \delta a_1^3 = 0, \\
(G')^1/G & : \quad -a_k c - 3a_k \lambda k^3 + \gamma a_1^3 + 3\delta a_1 a_1^2 = 0, \\
(G')^2/G & : \quad -a_k \lambda k^2 (2a_k \mu + a_k \lambda^3) + \mu a_1 + 2\gamma a_1 a_1 + 3\delta a_1 a_1 a_1 + \\
& \quad + \gamma (a_1^3 + 2a_k a_1 a_2 + \delta (a_1^3 + 6a_k a_1 a_2) = 0, \\
(G')^3/G & : \quad a_k \lambda k - 3a_k \lambda^2 k^3 + \gamma a_1^3 + 3\delta a_1 a_1^2 = 0, \\
(G')^4/G & : \quad -2a_k \mu k^2 + \delta a_1^3 = 0,
\end{align*}
\]

By solving the above set of algebraic equations with the aid of Maple or Mathematica we have the following cases:

**Case 1.**

\[
\lambda = 0, \quad \gamma = \gamma, \quad k = k, \quad \delta = \delta, \quad \mu = \frac{-1}{32k} (\gamma^2 - 4\delta \mu),
\]

\[
a_{i} = \frac{- (\gamma^2 - 4\delta \mu)}{32k \delta}, \quad a_{2} = k \frac{1}{\delta} c = \frac{k \gamma}{\sqrt{2 \delta}}, \quad a_{3} = \frac{- \gamma}{2 \delta}.
\]

**Case 2.**

\[
\lambda = \lambda, \quad \mu = \mu, \quad k = k, \quad \delta = \delta, \quad \mu = \frac{1}{4 \delta} (-k^2 \delta (\lambda^3 - 4\mu) + \gamma^3),
\]

\[
a_{1} = 0, \quad a_{2} = \frac{\gamma}{2 \delta}, \quad a_{3} = \frac{k \gamma}{\sqrt{2 \delta}}.
\]

Let us now write down the following exact solutions of the space-time fractional KPP equation (44) for case 1 (Similarly for case 2 which is omitted here for simplicity):

(i) If \( \mu < 0 \) (Hyperbolic function solutions)

In this case, we have the exact solution:

\[
u(x,t) = \frac{-2\mu}{\delta} \left( c \cosh \left( \sqrt{-\mu} \xi \right) + c \sinh \left( \sqrt{-\mu} \xi \right) \right) - \frac{\gamma}{2 \delta} \left( \gamma^2 - 4\delta \mu \right)^{-1} \cosh \left( \sqrt{-\mu} \xi \right).
\]

If we set \( c_1 = 0 \) and \( c_2 \neq 0 \) in (46) we have the solitary solution:

\[
u_s(x,t) = \frac{-2\mu}{\delta} \tanh \left( \sqrt{-\mu} \xi \right) - \frac{\gamma}{2 \delta} \left( \gamma^2 - 4\delta \mu \right)^{-1} \cosh \left( \sqrt{-\mu} \xi \right).
\]

while if we set \( c_2 = 0 \) and \( c_1 \neq 0 \), in (46) we have the solitary wave solution:
\[ u_1(x,t) = -k \sqrt{\frac{2\mu}{\delta}} \coth \left( \sqrt{-\mu \xi} \right) - \frac{\gamma}{2\delta} \]

\[ \frac{\left( \gamma^2 - 4\delta \mu \right)}{16k \delta \sqrt{2\mu \delta}} \tanh \left( \sqrt{-\mu \xi} + \xi_1 \right), \]

If \( \xi_1 \neq 0 \) and \( c_1^2 < c_1^2 \), then we have the solitary wave solution:

\[ u_1(x,t) = -k \sqrt{\frac{2\mu}{\delta}} \tan \left( \sqrt{-\mu \xi} + \xi_1 \right) - \frac{\gamma}{2\delta} \]

\[ \frac{\left( \gamma^2 - 4\delta \mu \right)}{16k \delta \sqrt{2\mu \delta}} \coth \left( \sqrt{-\mu \xi} + \xi_1 \right), \]

where \( \xi_1 = \coth^{-1} \left( \frac{c_2}{c_1} \right) \).

(ii) If \( \mu > 0 \) (Trigonometric function solutions)

In this case we have the exact solution:

\[ u(x,t) = -k \sqrt{\frac{2\mu}{\delta}} \left[ -c_1 \sin \left( \sqrt{\mu \xi} \right) + c_1 \cos \left( \sqrt{\mu \xi} \right) \right] - \frac{\gamma}{2\delta} \]

\[ - \frac{\left( \gamma^2 - 4\delta \mu \right)}{16k \delta \sqrt{2\mu \delta}} \left[ -c_1 \sin \left( \sqrt{\mu \xi} \right) + c_1 \cos \left( \sqrt{\mu \xi} \right) \right]^{-1} \cdot \left( c_2 \cos \left( \sqrt{\mu \xi} \right) + c_2 \sin \left( \sqrt{\mu \xi} \right) \right). \]

If we set \( c_1 = 0 \) and \( c_2 \neq 0 \) in (50) we have the periodic solution:

\[ u_1(x,t) = -k \sqrt{\frac{2\mu}{\delta}} \cot \left( \sqrt{\mu \xi} \right) - \frac{\gamma}{2\delta} \]

\[ \frac{\left( \gamma^2 - 4\delta \mu \right)}{16k \delta \sqrt{2\mu \delta}} \tan \left( \sqrt{\mu \xi} \right), \]

while if we set \( c_2 = 0 \) and \( c_1 \neq 0 \), in (50) we have the periodic solution:

**Figures of some exact solutions**

In this section we give some figures to illustrate the solutions of the equations (13), (24), (33) and (46) as follows:
Fig. 1 Numerical simulation solution of the space-time fractional PKP equation.

Fig. 2 Numerical simulation solution of the space-time fractional SRLW equation.
Fig. 3 Numerical simulation solution of the space-time fractional STO equation.
Some conclusions and discussions

In this article, we have extended successfully the improved \((G'/G)\)-expansion method to solve four nonlinear fractional partial differential equations. As applications, abundant new exact solutions for the space-time nonlinear fractional PKP equation, the space-time nonlinear fractional SRLW equation, the space-time nonlinear fractional STO equation and the space-time nonlinear fractional KPP equation have been successfully found. As one can see, the nonlinear fractional complex transformation for \(\xi\) is very important, which ensures that certain nonlinear fractional PDEs can be turned into other nonlinear ODEs of integer orders, whose solutions can be expressed in the form (8) where \(G(\xi)\) satisfies the linear ODE (9). Some numerical examples with diagrams have been given for fractional and non fractional orders to illustrate our results. Besides, as this method is based on the homogeneous balancing principle, so it can also be applied to other nonlinear fractional partial differential equations, where the homogeneous balancing principle is satisfied.

References


