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A NEW GENERATING FUNCTION OF CHEBYSHEV POLYNOMIALS

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ABSTRACT:

The object of the present paper is to point out that this class of generating relations implies the explicit representation, the addition and tile multiplication formulas, in addition to the usual generating relation for the Chebyshev polynomials.

Keywords: *Generating functions, Chebyshev polynomials.*

1. INTRODUCTION

An extension to the framework of the classical families of Laguerre, Jacobi, Hermite, Chebyshev and Gegenbauer polynomials have been introduced in [2, 3, 4, 5, 6, 12]. Orthogonal polynomials and special functions with integral representations have been considered in the books [1, 7, 8, 9, 10, 11, 13].

This paper deals with some generalizations of the Chebyshev polynomials of one variable defined by the following generating Functions

$$\sum_{n=0}^{\infty} U_n(x)t^n = \left(1-2xt+t^2\right)^{-1} \qquad (1.1)$$

The object of this note is to point out that one call make elementary verification of result (1.1) by means of any one of the following formulas of Chebyshev polynomials:

Explicit representation of the Chebyshev polynomials of the second kind are defined by the series

$$U_n(x) = \sum_{k=0}^{\left[\frac{1}{2}n\right]} \frac{(-1)^k (n-k)!(2x)^{n-2k}}{k!(n-2k)!}$$

(1.2)

The plane of the paper is the following. In section 2 generating functions involving the addition theorem of Chebyshev polynomials are derived and multiplication theorem is discussed and will provide the matter of forthcoming investigations.

2.Generating functions for Chebyshev polynomials.

(A) Additional theorem

$$\left(1+2xt+t^2\right)^{\frac{n}{2}} U_n\left(\frac{x+t}{\sqrt{1+2xt+t^2}}\right) = \sum_{k=0}^{n} \frac{(n+1)!}{(k+1)!(n-k)!} U_k(x) t^{n-k} \qquad (2.1)$$

For our purpose, we proceed to show that (1.2) is equivalent to (2.1). Putting

$$U = \frac{x}{\sqrt{1+2xt+t^2}},$$

$$v = \frac{t}{\sqrt{1+2xt+t^2}},$$

so that $\frac{u}{v} = \frac{x}{t}$ and therefore

$$x = \frac{u}{\sqrt{1-2uv-v^2}},$$

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$$t=\frac{v}{\sqrt{1-2uv-v^{2}}}$$

Thus the above addition theorem call be stated as follows

$$U_{n}(u+v)=\sum_{k=0}^{n}\frac{(n+1)!}{(k+1)!(n-k)!}\times$$

$$U_{k}\left(\frac{u}{\sqrt{1-2xt-t^{2}}}\right)\left(\frac{t}{\sqrt{1-2xt-t^{2}}}\right)^{n-k}.$$

(2.2)

(B) Multiplication Theorem

$$U_{n}(\mu x)=\sum_{k=0}^{\left[\frac{1}{2}n\right]}\mu^{n}\left(1-\frac{1}{\mu^{2}}\right)^{k}U_{n-2k}(x)$$

(2.3)

In other words, we wish to show that the generating function (1.1) contains many properties of Chebyshev polynomials, the explicit representation, the addition formula, the multiplication formula, in addition to the usual generating function (1.1).

For our purpose, we may write(1.1) in the form.

$$\left(1-2xt+t^{2}\right)^{-1}=\sum_{n=0}^{\infty}U_{n}\left(\mu x+\frac{1-\mu^{2}}{2\mu}t\right)\left(\frac{t}{\mu}\right)^{n}$$

(2.4)

First, we notice that

$$\left(1-2xt+t^{2}\right)^{-1}=\left(1+\frac{t^{2}}{\mu^{2}}\right)^{-1}\left[1-\frac{2xt\mu^{2}+t^{2}-\mu^{2}t^{2}}{\mu^{2}+t^{2}}\right]^{-1}$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^{k}(n+1)_{k}}{k!}\left(\frac{t^{2}}{\mu^{2}}\right)^{k}\sum_{n=0}^{\infty}\left[2xt+\frac{1-\mu^{2}}{\mu^{2}}t^{2}\right]^{n}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{k}(n+1)_{k}}{k!}\left(\frac{t}{\mu}\right)^{2k}\left[2xt+\frac{1-\mu^{2}}{\mu^{2}}t^{2}\right]^{n}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{k}(1)_{n+k}}{k!n!}\left(\frac{t}{\mu}\right)^{n+2k}\left[2\mu x+\frac{1-\mu^{2}}{\mu}t\right]^{n}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{k}(n+k)!}{k!n!}\left(\frac{t}{\mu}\right)^{n+2k}\left[2\mu x+\frac{1-\mu^{2}}{\mu}t\right]^{n}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{1}{2}n\right]}\frac{(-1)^{k}(n-k)!}{k!(n-2k)!}\left(\frac{t}{\mu}\right)^{n}\left[2\mu x+\frac{1-\mu^{2}}{\mu}t\right]^{n-2k}$$

$$=\sum_{n=0}^{\infty}\left(\frac{t}{\mu}\right)^{n}\sum_{k=0}^{\left[\frac{1}{2}n\right]}\frac{(-1)^{k}(n-k)!}{k!(n-2k)!}\left[2\mu x+\frac{1-\mu^{2}}{\mu}t\right]^{n-2k}$$

(2.5)

It follows therefore from (2.4) and (2.5) that

$$U_{n}\left(\mu x+\frac{1-\mu^{2}}{2\mu}t\right)=\sum_{k=0}^{\left[\frac{1}{2}n\right]}\frac{(-1)^{k}(n-k)!}{k!(n-2k)!}\times\left[2\mu x+\frac{1-\mu^{2}}{\mu}t\right]^{n-2k}$$

this is the explicit representation (1.2) for the Chebyshev polynomials.

Next, we have

$$\left(1-2xt+t^{2}\right)^{-1}=\left[\frac{(2\mu^{2}-(1-\mu^{2})t^{2})^{2}}{4\mu^{4}}\right]^{-1}\times\left[1-\frac{4xt\mu^{2}}{2\mu^{2}-(1-\mu^{2})t^{2}}+\frac{\zeta^{2}t^{2}}{(2\mu^{2}-(1-\mu^{2})t^{2})^{2}}\right]^{-1}$$

Where

$$\zeta=\left[4\mu^{2}-4xt\mu^{2}(1-\mu^{2})-(1-\mu^{2})^{2}t^{2}\right]^{\frac{1}{2}}$$

$$=\left[\frac{2\mu^{2}-(1-\mu^{2})t^{2}}{2\mu^{2}}\right]\sum_{k=0}^{\infty}\left(\frac{\zeta t}{2\mu^{2}-(1-\mu^{2})t^{2}}\right)^{k}\times U_{k}\left(\frac{2x\mu^{2}}{\zeta}\right)$$

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$$=\sum_{k=0}^{\infty}\left(\frac{\zeta t}{2\mu^2}\right)^k U_k\left(\frac{2x\mu^2}{\zeta}\right)\left[1-\frac{(1-\mu^2)t^2}{2\mu^2}\right]^{-(k+2)}$$

$$=\sum_{k=0}^{\infty}\left(\frac{\zeta t}{2\mu^2}\right)^k U_k\left(\frac{2x\mu^2}{\zeta}\right)$$

$$\times\sum_{n=0}^{\infty}\frac{(k+2)_n}{n!}\left[\frac{(1-\mu^2)t^2}{2\mu^2}\right]^n$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{(1-\mu^2)t^2}{2\mu^2}\right)^n\sum_{k=0}^{\infty}(k+2)_n$$

$$\times\left(\frac{\zeta t}{2\mu^2}\right)^k U_k\left(\frac{2x\mu^2}{\zeta}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{t}{\mu}\right)^{n+k}\frac{(k+2)_n\,\zeta^{n+k}}{n!(2\mu)^{n+k}}$$

$$\times\left(\frac{(1-\mu^2)t}{\zeta}\right)^n U_k\left(\frac{2x\mu^2}{\zeta}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{t}{\mu}\right)^{n+k}\frac{(2)_{n+k}\,\zeta^{n+k}}{n!(2)_k\,(2\mu)^{n+k}}$$

$$\times\left(\frac{(1-\mu^2)t}{\zeta}\right)^n U_k\left(\frac{2x\mu^2}{\zeta}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\frac{t}{\mu}\right)^{n}\frac{(2)_n\,\zeta^{n}}{(n-k)!(2)_k\,(2\mu)^{n}}$$

$$\times\left(\frac{(1-\mu^2)t}{\zeta}\right)^{n-k} U_k\left(\frac{2x\mu^2}{\zeta}\right). \quad (2.6)$$

It follows therefore from (2.4)and (2.6)that

$$U_n(\mu x+\frac{1-\mu^2}{2\mu}t)=\sum_{k=0}^{n}\frac{(2)_n\,\zeta^n}{(n-k)!(2)_k\,(2\mu)^n}$$

$$\times\left(\frac{(1-\mu^2)t}{\zeta}\right)^{n-k} U_k\left(\frac{2x\mu^2}{\zeta}\right).$$

this is the addition formula (2.2) for the Chebyshev polynomials, lastly we note that

$$\left(1-2xt+t^2\right)^{-1}=$$

$$\left[\frac{\left[2\mu^2-\left(1-\mu^2\right)^2t^2\right]^2-4xt\mu^2\left[4\mu^2-\left(1-\mu^2\right)t^2\right]+\zeta^2t^2}{4\mu^4}\right].$$

where

$$\zeta=\left[4\mu^2-4xt\mu^2\left(1-\mu^2\right)-\left(1-\mu^2\right)^2t^2\right]^{\frac{1}{2}}$$

$$=\left(1-\frac{\left(4\mu^4-\zeta^2\right)t^2}{R}\right)^{-1}\left(\frac{4\mu^4}{R}\right)$$

where

$$R=\left[2\mu^2-(1-\mu^2)t^2\right]^2$$
$$-4x\,t\,\mu^2\left[2\mu^2-(1-\mu^2)t^2\right]+4\mu^4t^2$$

$$=\left(\frac{4\mu^4}{R}\right)\sum_{k=0}^{\infty}\left(\frac{\left(4\mu^4-\zeta^2\right)t^2}{R}\right)^k$$

$$=\left(\frac{4\mu^4}{R}\right)\sum_{k=0}^{\infty}\left(\frac{\left(4\mu^4-\zeta^2\right)t^2}{4\mu^4}\right)^k\times$$
$$\left(\frac{2\mu^2-\left(1-\mu^2\right)t^2}{2\mu^2}\right)^{-2k-2}\left(\frac{R}{2\mu^2-\left(1-\mu^2\right)t^2}\right)^{-K-1}$$

$$=\left(\frac{4\mu^4}{R}\right)\sum_{k=0}^{\infty}\left(\frac{\left(4\mu^4-\zeta^2\right)t^2}{4\mu^4}\right)^k$$
$$\times\left(\frac{2\mu^2-\left(1-\mu^2\right)t^2}{2\mu^2}\right)^{-2k-2}$$

$$\times\sum_{m=0}^{\infty}U_m(x)\left(\frac{2t\mu^2}{2\mu^2-\left(1-\mu^2\right)t^2}\right)^m$$

$$=\sum_{m=0}^{\infty}U_m(x)t^m\left(\frac{4\mu^4}{R}\right)\sum_{k=0}^{\infty}\left(\frac{\left(4\mu^4-\zeta^2\right)t^2}{4\mu^4}\right)^k$$

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$$\times\left(\frac{2\mu^2-\left(1-\mu^2\right)t^2}{2\mu^2}\right)^{-2k-m-2}$$

$$=\sum_{m=0}^{\infty}U_m\left(x\right)t^m\left(\frac{4\mu^4}{R}\right)\sum_{k=0}^{\infty}\left(\frac{\left(4\mu^4-\zeta^2\right)t^2}{4\mu^4}\right)^k$$

$$\times\sum_{n=0}^{\infty}\frac{\left(2k+m+2\right)_n}{n!}\left(\frac{\left(1-\mu^2\right)t^2}{2\mu^2}\right)^n$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{n!}\left(\frac{\left(1-\mu^2\right)t^2}{2\mu^2}\right)^n\left(\frac{t\,\zeta}{2\mu^2}\right)^{m+2k}$$

$$\times\left(2k+m+2\right)_n\alpha^{m+2k}\left(1-\frac{1}{\alpha^2}\right)^k U_m\left(x\right)t^m$$

where

$$\alpha=\frac{2\mu^2}{\zeta}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\left(1-\mu^2\right)t^2}{2\mu^2}\right)^n\sum_{m=0}^{\infty}\left(\frac{\zeta t}{2\mu^2}\right)^m$$

$$\times\left(m+2\right)_n\sum_{k=0}^{\left[\frac{1}{2}m\right]}\alpha^m\left(1-\frac{1}{\alpha^2}\right)^k U_{m-2k}\left(x\right)\quad(2.7)$$

Now in driving (2.6) we have noticed

$$\left(1-2xt+t^2\right)^{-1}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\left(1-\mu^2\right)t^2}{2\mu^2}\right)^n$$

$$\times\sum_{m=0}^{\infty}\left(\frac{\zeta t}{2\mu^2}\right)^m\left(m+2\right)_n U_m\left(x\right)\quad(2.8)$$

It follows from (2.7) and (2.8) that

$$U_n\left(\alpha x\right)=\sum_{k=0}^{\left[\frac{1}{2}n\right]}\alpha^n\left(1-\frac{1}{\alpha^2}\right)^k U_{n-2k}\left(x\right)$$

this is the multiplication (2.3) for the Chebyshev polynomials and will provide the matter of forthcoming investigations.

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$$\left(2k+m+2\right)_n\alpha^{m+2k}\left(1-\frac{1}{\alpha^2}\right)U_m\left(x\right)t^m$$

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