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MOMENT GENERATING FUNCTION OF THE UNBALANCED NON-CENTRAL CHI-SQUARE DISTRIBUTION

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Abstract

In this paper, a general form of the cumulative distribution function and a moment generating function of the unbalanced (weighted) non-central chi-square distribution with ν degrees of freedom are obtained. Grau's result of moments (2009) can be obtained easily from the given moment generating function.

Key words: *Weighted non-central chi-square distribution, Capability indices, Moment generating function.*

1. Introduction

A random variable Z is defined to have a weighted non-central chi-square distribution with one degree of freedom and non-centrality parameter λ; $\chi_1^2(\lambda,\alpha_1,\alpha_2)$, if its probability density function was given as an infinite sum of central chi-square distributions as follows (see Chen (1998))

$$f_1(z)=\frac{e^{\frac{-\lambda}{2}}}{2\sqrt{\pi}}\sum_{j=0}^{\infty}\frac{(2\lambda)^{\frac{j}{2}}\,\Gamma\left(\frac{j+1}{2}\right)}{\Gamma(j+1)}\left[\alpha_1^{-2}\,f_{\chi^2_{1+j}}\left(\alpha_1^{-2}z\right)+(-1)^j\,\alpha_2^{-2}\,f_{\chi^2_{1+j}}\left(\alpha_2^{-2}z\right)\right],\ z>0,\lambda\ge 0 \qquad (1)$$

where α_1 and α_2 are two positive numbers and λ is the non-centrality parameter.

Grau (2009) generalized the $\chi_1^2(\lambda,\alpha_1,\alpha_2)$ distribution to the unbalanced (weighted) non-central chi-square distribution with ν degrees of freedom; $\chi_\nu^2(\lambda,\alpha_1,\alpha_2)$. Its probability density function was given as:

$$f_\nu(z)=\frac{e^{\frac{-\lambda}{2}}}{2\sqrt{\pi}}\sum_{j=0}^{\infty}\frac{(2\lambda)^{\frac{j}{2}}\,\Gamma\left(\frac{j+1}{2}\right)}{\Gamma(j+1)}\left[\alpha_1^{-2}\,f_{\chi^2_{\nu+j}}\left(\alpha_1^{-2}z\right)+(-1)^j\,\alpha_2^{-2}\,f_{\chi^2_{\nu+j}}\left(\alpha_2^{-2}z\right)\right],z>0,\lambda\ge 0 \qquad (2)$$

Grau (2009) obtained, for the $\chi_\nu^2(\lambda,\alpha_1,\alpha_2)$ distribution, the rth moments about zero and the central moments in specific cases. He expected that the knowledge of these moments could allow a different approach to obtain the moments of the estimators of the capability indices. Application of the unbalanced (weighted) non-central chi-square distribution with ν degrees of freedom may cover a wide spectrum of quality control areas. (Chan et al. (1988), Pearn et al. (1992), Greenwich and Jahr-Schaffrath (1995), Vannman (1995), Chen (1998), Pearn et al. (2001), Kotz and Johnson

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(2002), Montgomery (2005) and Kaya and Kahraman (2010)).

We intent to obtain a cumulative distribution function and a moment generating function of the $\chi_\nu^2(\lambda,\alpha_1,\alpha_2)$ distribution. In Section 2, we present the cumulative distribution function, while the moment generating function is given in Section 3. In Section 4, we derive from the moment generating function the first four moments about zero.

2. A cumulative distribution function

To obtain the cumulative distribution of the $\chi_\nu^2(\lambda,\alpha_1,\alpha_2)$ distribution rewrite the probability density function given in Equation (2) as follows:

$$f_\nu(z)=\sum_{k=0}^{\infty}c_k\left[\alpha_1^{-2}\,f_{\chi^2_{\nu+k}}\left(\alpha_1^{-2}z\right)+(-1)^k\,\alpha_2^{-2}\,f_{\chi^2_{\nu+k}}\left(\alpha_2^{-2}z\right)\right],$$

where

$$c_k=\frac{e^{\frac{-\lambda}{2}}}{2\sqrt{\pi}}\;\frac{(2\lambda)^{\frac{k}{2}}\,\Gamma\left(\frac{k+1}{2}\right)}{\Gamma(k+1)}\;.$$

Separating the odd and even terms of k, we get

$$f_\nu(z)=\sum_{j=0}^{\infty}c_{2j}\sum_{i=1}^{2}\alpha_i^{-2}f_{\chi^2_{\nu+2j}}\left(\alpha_i^{-2}z\right)+\sum_{j=0}^{\infty}c_{2j+1}\sum_{i=1}^{2}(-1)^{i+1}\alpha_i^{-2}f_{\chi^2_{\nu+2j+1}}\left(\alpha_i^{-2}z\right)$$

$$=\sum_{j=0}^{\infty}c_{2j}\,A+\sum_{j=0}^{\infty}c_{2j+1}\,B\;,\qquad z>0$$

where $A=\sum_{i=1}^{2}\alpha_i^{-2}\,f_{\chi^2_{\nu+2j}}\left(\alpha_i^{-2}\,z\right)$ and $B=\sum_{i=1}^{2}(-1)^{i+1}\alpha_i^{-2}\,f_{\chi^2_{\nu+2j+1}}\left(\alpha_i^{-2}\,z\right)$.

The cumulative distribution function is obtained as follows:

$$F_\nu(z)=\sum_{j=0}^{\infty}c_{2j}\int_0^z A\,dt\;+\;\sum_{j=0}^{\infty}c_{2j+1}\int_0^z B\,dt$$

Then

$$\int_0^z A\,dt=\sum_{i=1}^{2}\alpha_i^{-2}\int_0^z f_{\chi^2_{\nu+2j}}\left(\alpha_i^{-2}t\right)dt$$

$$=\sum_{i=1}^{2}\alpha_i^{-2}\int_0^z\frac{\left(\alpha_i^{-2}t\right)^{\frac{\nu+2j}{2}-1}}{2^{\frac{\nu+2j}{2}}\,\Gamma\left(\frac{\nu+2j}{2}\right)}e^{\frac{-\alpha_i^{-2}t}{2}}dt$$

$$=\frac{1}{\Gamma\left(\frac{\nu+2j}{2}\right)}\sum_{i=1}^{2}\gamma\left(\frac{\nu+2j}{2},\frac{\alpha_i^{-2}z}{2}\right)$$

where $\gamma(d,z)=\int_0^z x^{d-1}e^{-x}dx,$ is the lower incomplete gamma function . Similarly

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$$\int_0^z B\,dt = \frac{1}{\Gamma\left(\frac{\nu+2j+1}{2}\right)} \sum_{i=1}^{2} (-1)^{i+1} \gamma\left(\frac{\nu+2j+1}{2}, \frac{\alpha_i^{-2} z}{2}\right)$$

Finally, we have

$$F_\nu(z) = \frac{e^{\frac{-\lambda}{2}}}{2\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(2\lambda)^{\frac{k}{2}}\,\Gamma\left(\frac{k+1}{2}\right)}{\Gamma(k+1)\,\Gamma\left(\frac{\nu+k}{2}\right)} \left[\gamma\left(\frac{\nu+k}{2}, \frac{\alpha_1^{-2} z}{2}\right) + (-1)^k \gamma\left(\frac{\nu+k}{2}, \frac{\alpha_2^{-2} z}{2}\right)\right], \quad z>0,\ \lambda \ge 0 \quad (3)$$

For $\nu = 1$, Equation (3) becomes the cumulative distribution function of the unbalanced non-central chi-square distribution with one degree of freedom.

3. A moment generating function

To evaluate the moment generating function of the $\chi_\nu^2(\lambda, \alpha_1, \alpha_2)$ distribution, we need to prove the following lemma:

Lemma

$$\sum_{j=0}^{\infty} \frac{\left[\frac{\lambda}{2(1-2\alpha^2 t)}\right]^{\frac{2j+1}{2}}}{\Gamma\left(\frac{2j+3}{2}\right)} = e^{\frac{\lambda}{2(1-2\alpha^2 t)}}\, erf\left(\sqrt{\frac{\lambda}{2(1-2\alpha^2 t)}}\right), \quad (4)$$

where

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

Proof

In this proof we will use the following Laplace notations

1- $£\left[x^j\right] = \frac{\Gamma(j+1)}{P^{j+1}}, \quad P>0.$

2- $£\left[e^{ax}\, erf\left(\sqrt{a x}\right)\right] = \frac{\sqrt{a}}{\sqrt{P}\,(P-a)}, \quad P>0,\ a \text{ is a constant}.$

Using Laplace transform of the function in the L.H.S. of Equation (4) we get,

$$£\left[\sum_{j=0}^{\infty} \frac{\left(\frac{\lambda}{2(1-2\alpha^2 t)}\right)^{\frac{2j+1}{2}}}{\Gamma\left(\frac{2j+3}{2}\right)}\right] = \sum_{j=0}^{\infty} \frac{1}{\left(2(1-2\alpha^2 t)\right)^{\frac{2j+1}{2}}\,\Gamma\left(\frac{2j+3}{2}\right)}\, £\left[\lambda^{\frac{2j+1}{2}}\right]$$

Using Notation 1, we get

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$$\pounds\left[\sum_{j=0}^{\infty}\frac{\left(\frac{\lambda}{2(1-2\alpha^2 t)}\right)^{\frac{2j+1}{2}}}{\Gamma\left(\frac{2j+3}{2}\right)}\right]=\frac{1}{\sqrt{2(1-2\alpha^2 t)}}\sum_{j=0}^{\infty}\frac{1}{[2(1-2\alpha^2 t)]^j\, P^{\frac{2j+3}{2}}}$$

$$=\frac{1}{\sqrt{2(1-2\alpha^2 t)}}\,\frac{1}{\sqrt{P}\left(P-\frac{1}{2(1-2\alpha^2 t)}\right)}$$

Using Notation 2, we get

$$\pounds\left[\sum_{j=0}^{\infty}\frac{\left(\frac{\lambda}{2(1-2\alpha^2 t)}\right)^{\frac{2j+1}{2}}}{\Gamma\left(\frac{2j+3}{2}\right)}\right]=\pounds\left[e^{\frac{\lambda}{2(1-2\alpha^2 t)}}\,erf\left(\sqrt{\frac{\lambda}{2(1-2\alpha^2 t)}}\right)\right]$$

Then the result of Equation (4) □

The moment generating function of the $\chi_\nu^2(\lambda,\alpha_1,\alpha_2)$ distribution with density function given by Equation (2) is obtained as follows:

$$M_Z(t)=E\left(e^{tZ}\right)=\int_0^{\infty}e^{tz}\,f_\nu(z)\,dz$$

$$=\int_0^{\infty}e^{tz}\left[\sum_{j=0}^{\infty}c_{2j}\sum_{i=1}^{2}\alpha_i^{-2}\,f_{\chi^2_{\nu+2j}}\left(\alpha_i^{-2}z\right)+\sum_{j=0}^{\infty}c_{2j+1}\sum_{i=1}^{2}(-1)^{i+1}\alpha_i^{-2}\,f_{\chi^2_{\nu+2j+1}}\left(\alpha_i^{-2}z\right)\right]dz$$

$$M_Z(t)=\sum_{j=0}^{\infty}\frac{c_{2j}}{2^{\frac{\nu+2j}{2}}\Gamma\left(\frac{\nu+2j}{2}\right)}\int_0^{\infty}e^{tz}\sum_{i=1}^{2}\frac{z^{\frac{\nu+2j}{2}-1}e^{\frac{-\alpha_i^{-2}z}{2}}}{\alpha_i^{\nu+2j}}\,dz$$

$$+\sum_{j=0}^{\infty}\frac{c_{2j+1}}{2^{\frac{\nu+2j+1}{2}}\Gamma\left(\frac{\nu+2j+1}{2}\right)}\int_0^{\infty}e^{tz}\sum_{i=1}^{2}\frac{z^{\frac{\nu+2j+1}{2}-1}e^{\frac{-\alpha_i^{-2}z}{2}}}{\alpha_i^{\nu+2j+1}}\,dz$$

Using the formula, (see Gradshteyn and Ryzhik (1980)),

$$\int_0^{\infty}x^{\nu-1}\,e^{-\mu x}\,dx=\frac{\Gamma(\nu)}{\mu^{\nu}}\ ,\ \mu,\nu>0,$$

we get

$$M_Z(t)=\sum_{j=0}^{\infty}c_{2j}\sum_{i=1}^{2}\left(1-2\alpha_i^2 t\right)^{\frac{-(\nu+2j)}{2}}+\sum_{j=0}^{\infty}c_{2j+1}\sum_{i=1}^{2}(-1)^{i+1}\left(1-2\alpha_i^2 t\right)^{\frac{-(\nu+2j+1)}{2}}$$

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Therefore,

$$M_Z(t)=\frac{e^{\frac{-\lambda}{2}}}{2\sqrt{\pi}}\sum_{j=0}^{\infty}\frac{(2\lambda)^{\frac{2j}{2}}\Gamma\left(\frac{2j+1}{2}\right)}{\Gamma(2j+1)}\sum_{i=1}^{2}\left(1-2\alpha_i^2 t\right)^{\frac{-(\nu+2j)}{2}}$$
$$+\frac{e^{\frac{-\lambda}{2}}}{2\sqrt{\pi}}\sum_{j=0}^{\infty}\frac{(2\lambda)^{\frac{2j+1}{2}}\Gamma\left(\frac{2j+2}{2}\right)}{\Gamma(2j+2)}\sum_{i=1}^{2}(-1)^{i+1}\left(1-2\alpha_i^2 t\right)^{\frac{-(\nu+2j+1)}{2}} \quad (5)$$

Since (see Kettani (2006)),

$$\Gamma\left(j+\frac{1}{2}\right)=\frac{\Gamma(2j+1)\sqrt{\pi}}{2^{2j}\,\Gamma(j+1)} \quad , \; j\geq 0$$

Consequently,

$$\Gamma(j+1)=\frac{\Gamma(2j+2)\sqrt{\pi}}{2^{2j+1}\,\Gamma\left(j+\frac{3}{2}\right)} \quad , \; j\geq 0$$

Then Equation (5) becomes

$$M_Z(t)=\frac{e^{\frac{-\lambda}{2}}}{2}\left[\sum_{j=0}^{\infty}\frac{(\lambda/2)^j}{\Gamma(j+1)}\sum_{i=1}^{2}\left(1-2\alpha_i^2 t\right)^{\frac{-(\nu+2j)}{2}}+\sum_{j=0}^{\infty}\frac{(\lambda/2)^{\frac{2j+1}{2}}}{\Gamma\left(\frac{2j+3}{2}\right)}\sum_{i=1}^{2}(-1)^{i+1}\left(1-2\alpha_i^2 t\right)^{\frac{-(\nu+2j+1)}{2}}\right]$$
$$=\frac{e^{\frac{-\lambda}{2}}}{2}\left[\sum_{i=1}^{2}\left(1-2\alpha_i^2 t\right)^{\frac{-\nu}{2}}e^{\frac{\lambda}{2(1-2\alpha_i^2 t)}}+\sum_{i=1}^{2}(-1)^{i+1}\left(1-2\alpha_i^2 t\right)^{\frac{-\nu}{2}}C_i\right],$$

where

$$C_i=\sum_{j=0}^{\infty}\frac{\left(\frac{\lambda}{2\left(1-2\alpha_i^2 t\right)}\right)^{\frac{2j+1}{2}}}{\Gamma\left(\frac{2j+3}{2}\right)} \quad , \; i=1,2$$

Using Equation (4) with $\alpha=\alpha_i\;, i=1,2$ to evaluate C_i, we get the moment generating function of the unbalanced (weighted) non-central chi-square distribution, i.e.,

$$M_Z(t)=\frac{1}{2}\sum_{i=1}^{2}e^{\frac{\lambda\alpha_i^2 t}{\left(1-2\alpha_i^2 t\right)}}\left(1-2\alpha_i^2 t\right)^{-\nu/2}\left\{1+(-1)^{i+1}erf\left(\sqrt{\frac{\lambda}{2\left(1-2\alpha_i^2 t\right)}}\right)\right\},\; for\; t<\frac{1}{2\alpha_i^2}, i=1,2 \quad (6)$$

From Equation (6), we can derive the following special cases:

(i) For $\alpha_1=\alpha_2=1$, Equation (6) reduced to the moment generating function of the non-central chi-square distribution $\chi_\nu^2(\lambda)$, (see Johnson et al. (1994)) i.e.,

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$$M_Z(t)=\frac{1}{2}\sum_{i=1}^{2} e^{\frac{\lambda t}{(1-2t)}}(1-2t)^{-\nu/2}\left\{1+(-1)^{i+1} erf\left(\sqrt{\frac{\lambda}{2(1-2t)}}\right)\right\}$$

$$=(1-2t)^{-\nu/2} e^{\frac{\lambda t}{(1-2t)}} \quad for \quad t<\frac{1}{2}$$

(ii) For $\lambda=0$, Equation (6) reduced to the moment generating function of the unbalanced central chi-square distribution; $\chi_\nu^2(0,\alpha_1,\alpha_2)$; i.e.,

$$M_Z(t)=\frac{1}{2}\sum_{i=1}^{2}(1-2\alpha_i^2 t)^{-\nu/2} \quad for \quad t<\frac{1}{2\alpha_i^2},\ i=1,2$$

(iii) For $\lambda=0,\ \alpha_1=\alpha_2=1$, Equation (6) reduced to the moment generating function of the central chi-square distribution χ_ν^2, (see Johnson et al. (1994) i.e.,

$$M_Z(t)=(1-2t)^{\frac{-\nu}{2}} \qquad for \quad t<\frac{1}{2}$$

4. Moments

To obtain the r^th moment about zero; $r=1,2,3,4$, we differentiate the moment generating function of Equation (6) r times then we put $t=0$ (for the details you can contact with the authors) and we get:

$$\mu_1^{'}=E(Z)=\frac{1}{2}(\alpha_1^2+\alpha_2^2)(\nu+\lambda)+\frac{1}{2}(\alpha_1^2-\alpha_2^2)\left[\sqrt{\frac{2\lambda}{\pi}}\, e^{\frac{-\lambda}{2}}+(\nu+\lambda)\, erf\left(\sqrt{\frac{\lambda}{2}}\right)\right],$$

$$\mu_2^{'}=E(Z^2)=\frac{1}{2}(\alpha_1^4+\alpha_2^4)\left[\nu(\nu+2)+2(\nu+2)\lambda+\lambda^2\right]+\frac{1}{2}(\alpha_1^4-\alpha_2^4).$$
$$\left[(\lambda+(2\nu+3))\sqrt{\frac{2\lambda}{\pi}}\, e^{\frac{-\lambda}{2}}+\left(\nu(\nu+2)+2(\nu+2)\lambda+\lambda^2\right) erf\left(\sqrt{\frac{\lambda}{2}}\right)\right],$$

$$\mu_3^{'}=E(Z^3)=\frac{1}{2}(\alpha_1^6+\alpha_2^6)\left[\sum_{j=0}^{3} C_j^3\,\lambda^j \prod_{i=j}^{2}(\nu+2i)\right]+\frac{1}{2}(\alpha_1^6-\alpha_2^6).$$
$$\left[\left(\lambda^2+(3\nu+11)\lambda+3(\nu^2+5\nu+5)\right)\sqrt{\frac{2\lambda}{\pi}}\, e^{\frac{-\lambda}{2}}+\left(\sum_{j=0}^{3} C_j^3\,\lambda^j\prod_{i=j}^{2}(\nu+2i)\right) erf\left(\sqrt{\frac{\lambda}{2}}\right)\right],$$

and

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$$\mu_4' = E(Z^4) = \frac{1}{2}\left(\alpha_1^8+\alpha_2^8\right)\left[\sum_{j=0}^{4} C_j^4\,\lambda^j \prod_{i=j}^{3}(\nu+2i)\right] + \frac{1}{2}\left(\alpha_1^8-\alpha_2^8\right).$$

$$\left[\begin{array}{l}\left(\lambda^3+(4\nu+23)\lambda^2+\left(6\nu^2+56\nu+123\right)\lambda+\left(4\nu^3+42\nu^2+128\nu+105\right)\right)\sqrt{\frac{2\lambda}{\pi}}\,e^{\frac{-\lambda}{2}}+\\ \left(\sum_{j=0}^{4} C_j^4\,\lambda^j \prod_{i=j}^{3}(\nu+2i)\right) erf\left(\sqrt{\frac{\lambda}{2}}\right)\end{array}\right],$$

It is clear that these results of the four moments about zero are the same as that obtained by Grau (2009).

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