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ON THE EIGENFUNCTION EXPANSION METHOD FOR THE CALCULATION OF GREEN'S FUNCTIONS

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ABSTRACT

In this work, we discuss some aspects of the eigenfunction expansion method for the calculation of Green's functions. To this end, we completely solve Poisson's equation in a bounded domain under the Dirichlet boundary conditions. The problem geometry and boundary data were chosen so as to show up the features investigated. The systematic and detailed calculation presented here is an attempt to fill up a gap in the literature.

Keywords: *Eigenfunction expansion, Green's function, Poisson's equation.*

INTRODUCTION

Green's functions are often determined as an expansion in eigenfunctions. However, depending on the number of spatial dimensions, two or more sets of orthogonal eigenfunctions may become available. Therefore, an issue that surely arises is that of deciding which of the sets is the more appropriate. This work addresses this question.

To this end, we consider a specific problem. We solve Poisson's equation in a half-disk under non-homogeneous Dirichlet boundary conditions employing, as a natural choice, the plane polar coordinates. We will see that the choice of the eigenfunctions is of special importance for problems with non-homogeneous boundary conditions.

That particular problem also enables the discussion of three special features that the eigenfunction expansion method may exhibit. The first special feature is an uncommon eigenvalue problem, whose spectrum is continuous, in spite of the bounded domain considered. The second one refers to the possibility of expressing the resulting Green's function expansion in a closed form (that is, in terms of the elementary functions). The third feature is the possibility of performing the integral that yields Green's function as an expansion in the continuous-spectrum eigenfunctions.

In the literature, it is hard to find the Green's function method applied to problems having non-homogeneous boundary conditions. As an attempt to fill up this gap, the calculations are presented systematically and in detail.

Section 2 contains the formulation of the problem considered and of its solution in terms of Green's function. Section 3 presents the calculation of Green's function as expansions in two kinds of eigenfunctions. Section 4 describes the determination of the problem solution in terms of the Green's functions calculated in Section 3. Section 5 concludes the body of the paper with a discussion of the results.

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2. FORMULATION OF THE PROBLEM AND OF THE SOLUTION

It is well established (References [1, Chap. 5], [2, Sec. 12.8], [3, Sec. 11.9], [4, Sec. 1.10] and [5, Sec. 7.2]) that the solution of Poisson's equation under the Dirichlet boundary condition,

$$\nabla^2\psi(\boldsymbol{\rho}) = -4\pi f(\boldsymbol{\rho}) \quad [\boldsymbol{\rho}\in\mathcal{A}] \;, \qquad (1)$$

$$\psi(\boldsymbol{\rho}) = g(\boldsymbol{\rho}) \quad [\boldsymbol{\rho}\in\partial\mathcal{A}] \;, \qquad (2)$$

where $\mathcal{A}$ is a domain of the *xy*-plane and $\partial\mathcal{A}$ is its boundary, is given by

$$\psi(\boldsymbol{\rho}) = \int_{\mathcal{A}} dA'\, G(\boldsymbol{\rho}\,|\,\boldsymbol{\rho}')\, f(\boldsymbol{\rho}') - \frac{1}{4\pi}\oint_{\partial\mathcal{A}} ds'\,\frac{\partial G}{\partial n'}(\boldsymbol{\rho}\,|\,\boldsymbol{\rho}')\, g(\boldsymbol{\rho}') \;, \qquad (3)$$

where $G(\boldsymbol{\rho}|\boldsymbol{\rho}')$ $[\boldsymbol{\rho}'\in\mathcal{A}]$ is the solution of

$$\nabla^2 G(\boldsymbol{\rho}\,|\,\boldsymbol{\rho}') = -4\pi\,\delta(\boldsymbol{\rho}-\boldsymbol{\rho}') \quad [\boldsymbol{\rho}\in\mathcal{A}] \;, \qquad (4)$$

$$G(\boldsymbol{\rho}\,|\,\boldsymbol{\rho}') = 0 \quad [\boldsymbol{\rho}\in\partial\mathcal{A}] \;, \qquad (5)$$

being $\partial G/\partial n$ the normal derivative, equal to $\boldsymbol{n}\cdot\nabla G$, with the unit normal vector $\boldsymbol{n}$ directed outward from $\mathcal{A}$ at a point of $\partial\mathcal{A}$.

Let $\mathcal{A}$ be the half-disk shown in Figure 1. In this geometry, the plane polar coordinates ρ and φ are the most suitable. For the boundary conditions given in that figure, (1) and (2) become

$$\nabla^2\psi(\rho,\varphi) = -4\pi f(\rho,\varphi) \;, \qquad (6)$$

$$\psi(1,\varphi) = g_1(\varphi) \;, \qquad (7)$$

$$\psi(\rho,0) = 0 \,, \quad \psi(\rho,\pi) = g_\pi(\varphi) \;, \qquad (8)$$

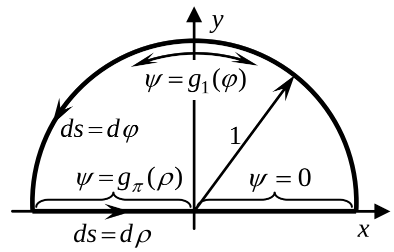

Figure 1 – Specification of the domain $\mathcal{A}$ and of the boundary data $g(\varphi)$ for the problem defined by (1) and (2).

with $0\le\rho\le1$ and $0\le\varphi\le\pi$. Likewise, Equations (4) and (5) now read

$$\frac{\partial^2 G}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial G}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 G}{\partial\varphi^2}(\rho,\varphi\,|\,\rho',\varphi') = -(4\pi/\rho)\,\delta(\rho-\rho')\,\delta(\varphi-\varphi') \;, \qquad (9)$$

$$G(1,\varphi\,|\,\rho',\varphi') = 0 \;, \qquad (10)$$

$$G(\rho,0\,|\,\rho',\varphi') = G(\rho,\pi\,|\,\rho',\varphi') = 0 \;, \qquad (11)$$

with ρ' and φ' varying as ρ and φ vary. The solution of our problem [the problem defined by (6) to (8)] is, in accordance with (3),

$$\psi(\rho,\varphi) \equiv \psi_f(\rho,\varphi) + \psi_1(\rho,\varphi) + \psi_\pi(\rho,\varphi) \;, \qquad (12)$$

where the source term is given by

$$\psi_f = \int_0^\pi\!\!\int_0^1 G(\rho,\varphi\,|\,\rho',\varphi')\, f(\rho',\varphi')\,\rho'\,d\rho'\,d\varphi' \;, \qquad (13)$$

and the boundary terms, by

$$\psi_1(\rho,\varphi) = \frac{-1}{4\pi}\int_0^\pi \frac{\partial G}{\partial\rho'}(\rho,\varphi\,|\,1,\varphi')\, g_1(\varphi')\,d\varphi' \;, \qquad (14)$$

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$$\psi_\pi(\rho,\varphi)=\frac{-1}{4\pi}\int_0^1 \frac{\partial G}{\partial\varphi'}(\rho,\varphi|\rho',\pi)\, g_\pi(\rho')\frac{d\rho'}{\rho'} \quad . \tag{15}$$

3. CALCULATION OF GREEN'S FUNCTION

To calculate G, we consider two subregions of $\mathcal{A}$, those obtained with either a radial division, $\rho<\rho'$ and $\rho>\rho'$, or a sectorial division, $\varphi<\varphi'$ and $\varphi>\varphi'$ (Figure 2 below). Let us consider the radial division first. In each subregion $\rho\neq\rho'$, we have that and, therefore, $\delta(\rho-\rho')$ vanishes; that is, the PDE for G given by (9) is homogeneous and can be solved by means of the method of separation of variables. Thus, substituting $G\equiv R(\rho)F(\varphi)$, we obtain

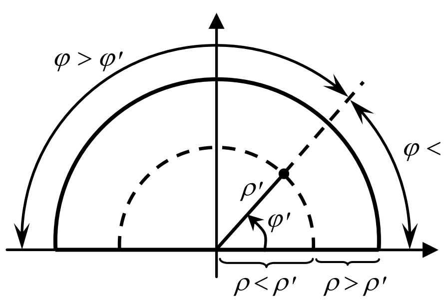

Figure 2 – The two ways the problem domain is divided in two subregions: radially ($\rho<\rho'$, $\rho>\rho'$) and sectorially ($\varphi<\varphi'$, $\varphi>\varphi'$).

$$\frac{\rho^2R''+\rho R'}{R}+\underbrace{\frac{F''}{F}}_{-\mu}=0 \quad , \tag{16}$$

where we have recognized that the second term must be a constant, $-\mu$. The separated ODE $F''+\mu F(\varphi)=0$ is to be solved in each subregion under the boundary conditions $F(0)=F(\pi)=0$ [derived from (11)]. This problem (with homogeneous ODE and boundary conditions) is an eigenvalue problem, whose solutions are easily found to consist of the eigenvalues $\mu_n=n^2$ ($n=1,2,3\ldots$) and the eigenfunctions $F_n(\varphi)=\sin n\varphi$ (Reference [2, Sec. 8.2]). [Notice that the radial problem, formed with the separated ODE for $R(\rho)$, is not an eigenvalue problem; in fact, on the common boundary at $\rho=\rho'$ of both subregions, *homogeneous* conditions cannot be derived!]

The resulting eigenfunctions can be used to express Green's function as a linear superposition of terms of the type $R_n(\rho)F_n(\rho)=R_n(\rho)\sin n\varphi$:

$$G(\rho,\varphi\,|\,\rho',\varphi')=\sum_{n=1}^{\infty}R_n(\rho)\sin n\varphi \quad . \tag{17}$$

To determine the functions $R_n(\rho)$ (whose dependence on ρ' and φ' is implicit), we substitute the above expansion into the PDE given by (9), obtaining

$$\sum_{n=1}^{\infty}\left[R_n''+\rho^{-1}R_n'-(n/\rho)^2R_n(\rho)\right]\sin n\varphi$$
$$=-\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\varphi-\varphi') \quad , \tag{18}$$

from which we infer that the term enclosed by brackets are the coefficients of the Fourier sine series of the function on the right-hand side over the interval $(0,\pi)$, that is,

$$R_n''+\rho^{-1}R_n'-\left(\frac{n}{\rho}\right)^2R_n(\rho)=$$
$$\frac{2}{\pi}\int_0^\pi\left[\frac{-4\pi}{\rho}\delta(\rho-\rho')\delta(\varphi-\varphi')\right]\sin n\varphi\, d\varphi$$
$$=-\frac{8}{\rho}\sin n\varphi'\,\delta(\rho-\rho') \quad . \tag{19}$$

This is an Euler equation, which is homogeneous in each subregion (where $\rho\neq\rho'$). Its solution is (Reference [3, Sec. 1.6])

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$$R_n(\rho) = \begin{cases} A_{1n}\,\rho^n + B_{1n}\rho^{-n} & (\rho < \rho') \\ A_{2n}\,\rho^n + B_{2n}\rho^{-n} & (\rho > \rho') \end{cases} \qquad (20)$$

The four constants above are determined by imposing the following four conditions (Reference [2, Sec. 12.2]):

(i) Finiteness at the origin, which is achieved by setting $B_{1n} = 0$.

(ii) The condition $R_n(1) = A_{2n} + B_{2n} = 0$ which follows from (10).

(iii) The continuity condition $R(\rho'^{+}) = R(\rho'^{-})$ at $\rho = \rho'$, since $G = RF$ is a potential and, therefore, must be a continuous function. Observe the notation $\rho'^{\pm} \equiv \rho' \pm \varepsilon$, with $\varepsilon \to 0^{+}$.

(iv) The jump discontinuity condition for the derivative of $R(\rho)$,

$$R'(\rho'^{+}) - R'(\rho'^{-}) = -\frac{8}{\rho'}\sin n\varphi' \,,$$

obtained by integrating (19) in the neighborhood of ρ', from ρ'^{-} to ρ'^{+}.

Once the calculation of (20) is completed, its substitution into (17) yields

$$G^{\frown}(\rho,\varphi\,|\,\rho',\varphi') =$$

$$\sum_{n=1}^{\infty} \frac{-4}{n}\rho_<^n\,(\rho_>^n - \rho_>^{-n})\,\sin n\varphi' \sin n\varphi \,, \qquad (21)$$

where $\rho_<\,(\rho_>)$ is the smaller (larger) of ρ and ρ'. The symbol $^{\frown}$ – resembling the radial division of $\mathcal{A}$ – is used to indicate the calculation of Green's function considering this division of $\mathcal{A}$.

Let us calculate Green's function again, this time considering the region $\mathcal{A}$ divided in the two sectors $\varphi < \varphi'$ and $\varphi > \varphi'$. In both of them, it is now for $R(\rho)$ that an eigenvalue problem arises with the separation of variables $G \equiv R(\rho)\,F(\varphi)$, because of the homogeneous condition $R(1) = 0$ [deduced from (10)] on the boundary at $\rho = 1$ of both sectors. We thus separate the ODE $\rho^2 R'' + \rho R' + \lambda R(\rho) = 0$ by equating the first term in (16) to the constant $-\lambda$.

The eigenvalue problem so obtained can be converted to a familiar one by changing the independent variable to $u \equiv -\ln\rho$. It becomes $\ddot{\bar{R}} + \lambda\bar{R}(u) = 0$, with $\bar{R}(0) = 0$ and $u \geq 0$, where $\bar{R}(u) \equiv R[\rho(u)]$ and $\rho(u) = e^{-u}$. The well-known eigenvalues and eigenfunctions are $\lambda_k = k^2$ and $\bar{R}_k(u) = \sin ku$ [or $R_k(\rho) = \sin(\ln\rho)$], with $k > 0$ (a continuous spectrum: cf. Reference [2, Sec. 8.7]). In many instances, it is better to work with the new variable *u*, in terms of which (9) reads

$$\frac{\partial^2\bar{G}}{\partial u^2} + \frac{\partial^2\bar{G}}{\partial\varphi^2} = -4\pi\,\delta(u-u')\,\delta(\varphi-\varphi') \,, \qquad (22)$$

where $\bar{G}(u,\varphi\,|\,u',\varphi') \equiv G[\rho(u),\varphi\,|\,\rho'(u'),\varphi']$. The calculation of $\bar{G}$ proceeds in the same manner described above. We substitute the expansion

$$\bar{G}(u,\varphi\,|\,u',\varphi') = \int_0^{\infty} F_k(\varphi)\sin ku\,du \qquad (23)$$

into (22) to obtain

$$\int_0^{\infty} [\,F_k'' - k^2 F_k(\varphi)\,]\sin ku\,du$$

$$= -4\pi\,\delta(u-u')\,\delta(\varphi-\varphi') \,.$$

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Then, by using the Fourier sine integral formula (Reference [6, Sec. 64]), we calculate the term in the integrand which is enclosed by brackets, obtaining the equation

$$F_k'' - k^2 F_k(\varphi) = \frac{2}{\pi}\int_0^\infty \left[-4\pi\,\delta(u-u')\,\delta(\varphi-\varphi')\right]\sin ku\,du = -8\sin ku'\,\delta(\varphi-\varphi') \ . \qquad (24)$$

Next, we solve it separately in each sector,

$$F_k(\varphi) = \begin{cases} A_{1k}\cosh k\varphi + B_{1k}\sinh k\varphi & (\varphi<\varphi') \\ A_{2k}\cosh k\varphi + B_{2k}\sinh k\varphi & (\varphi>\varphi') \end{cases}$$

and determine the four constants by imposing the four conditions: (i) $F_k(0)=0$ and (ii) $F_k(\pi)=0$ [both from the boundary condition (11)]; (iii) $F_k(\varphi'^{+}) = F_k(\varphi'^{-})$ (continuity at $\varphi=\varphi'$); (iv) $F_k'(\varphi'^{+}) - F_k'(\varphi'^{-}) = -8\sin ku'$ (jump discontinuity of $F_k'(\varphi)$ at $\varphi=\varphi'$, derived by integrating (24) in the neighborhood of φ', from φ'^{-} to φ'^{+}).

Finally, we substitute the $F_k(\varphi)$ so determined into (23) to obtain

$$\bar{G}^{\vee}(u,\varphi\,|\,u',\varphi') = \int_0^\infty \frac{8\sin ku\sin ku'\sinh k\varphi_<\,\sinh k(\pi-\varphi_>)\,dk}{k\,\sinh k\pi} \ , \qquad (25)$$

where $\varphi_<$ ($\varphi_>$) is the smaller (larger) of φ and φ'. The symbol $^{\vee}$ (like a sector) indicates that G is calculated considering the problem domain $\mathcal{A}$ divided into two sectors.

4. THE SOLUTION IN TERMS OF THE CALCULATED GREEN'S FUNCTION

We develop below only the boundary terms $\psi_1(\rho,\varphi)$ and $\psi_\pi(\rho,\varphi)$ appearing in (12); the source term $\psi_f(\rho,\varphi)$ is pretty well discussed in the literature (e.g., Reference [4]).

Looking at (14) and (15), we see that we need to calculate $\partial G/\partial\rho'$ at $\rho'=1$ and $\partial G/\partial\varphi'$ at $\varphi'=\pi$. Using (21) first, we obtain

$$\frac{\partial G^{\frown}}{\partial\rho'}(\rho,\varphi\,|\,1,\varphi') = \frac{\partial}{\partial\rho'}\sum_{n=1}^\infty \frac{-4}{n}\rho^n(\rho'^{n}-\rho'^{-n})\sin n\varphi'\sin n\varphi\Big|_{\rho'=1} = -8\sum_{n=1}^\infty \rho^n\sin n\varphi'\sin n\varphi$$

and

$$\frac{\partial G^{\frown}}{\partial\varphi'}(\rho,\varphi|\rho',\pi) = \frac{\partial}{\partial\varphi'}\sum_{n=1}^\infty \frac{-4}{n}\rho_<^n(\rho_>^n-\rho_>^{-n})\sin n\varphi'\sin n\varphi\Big|_{\varphi'=\pi} = -4\sum_{n=1}^\infty (-1)^n\rho_<^n(\rho_>^n-\rho_>^{-n})\sin n\varphi \ .$$

Substituting these results into (14) and (15) , we obtain

$$\psi_1^{\frown}(\rho,\varphi) = \sum_{n=1}^\infty \gamma_{1n}\,\rho^n\sin n\varphi \ , \qquad (26)$$

$$\psi_\pi^{\frown}(\rho,\varphi) \equiv \sum_{n=1}^\infty (-1)^n I_{\pi n}(\rho)\sin n\varphi \ , \qquad (27)$$

where

$$\gamma_{1n} \equiv \frac{2}{\pi}\int_0^\pi g_1(\varphi')\sin n\varphi'\,d\varphi' \ , \qquad (28)$$

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$$I_{\pi n}(\rho) \equiv \frac{1}{\pi}\int_0^1 g_\pi(\rho')\,\rho_<^n\,(\rho_>^n - \rho_>^{-n})\frac{d\rho'}{\rho'} \ .$$

Now using (25) to calculate $\partial G/\partial\rho'$ at $\rho'=1$ and $\partial G/\partial\varphi'$ at $\varphi'=\pi$, we get

$$\frac{\partial G^\vee}{\partial\rho'}(\rho,\varphi\,|\,1,\varphi') = \left[-e^{u'}\frac{\partial}{\partial u'}\int_0^\infty dk \; \frac{8\sin ku \sin ku' \sinh k\varphi_< \sinh k(\pi-\varphi_>)}{k\,\sinh k\pi}\right]_{u'=0}$$

$$= -8\int_0^\infty \frac{\sin ku \sinh k\varphi_< \sinh k(\pi-\varphi_>)}{\sinh k\pi}dk \ .$$

and

$$\frac{\partial G^\vee}{\partial\varphi'}(\rho,\varphi\,|\,\rho',\pi) = \left[\frac{\partial}{\partial\varphi'}\int_0^\infty dk \; \frac{8\sin ku \sin ku' \sinh k\varphi \sinh k(\pi-\varphi')}{k\,\sinh k\pi}\right]_{\varphi'=\pi}$$

$$= -8\int_0^\infty \frac{\sin ku \sin ku' \sinh k\varphi}{\sinh k\pi}dk \ .$$

Substitution into (14) and (15) gives

$$\psi_1^\vee(\rho,\varphi) \equiv \int_0^\infty dk\, I_{1k}(\varphi)\frac{\sin(k\ln\rho)}{\sinh k\pi} \ , \qquad (29)$$

$$\psi_\pi^\vee(\rho,\varphi) \equiv -\int_0^\infty dk\,\gamma_\pi(k)\frac{\sinh k\varphi \sin(k\ln\rho)}{\sinh k\pi} \ , \qquad (30)$$

where

$$I_{1k}(\varphi) \equiv \frac{2}{\pi}\int_0^\pi d\varphi' g_1(\varphi')\sinh k\varphi_< \sinh k(\pi-\varphi_>) \ ,$$

$$\gamma_\pi(k) \equiv \frac{2}{\pi}\int_0^\infty du'\,\bar{g}_\pi(u')\sin ku' \ . \qquad (31)$$

5. DISCUSSION OF THE RESULTS

a) The solution $\psi(\rho,\varphi)$ of our problem [defined by (6), (7) and (8)] is split in three terms, according to (12). We discuss here the expressions obtained for the boundary terms $\psi_1(\rho,\varphi)$ and $\psi_\pi(\rho,\varphi)$. More specifically, we verify the ability of these terms to reproduce the boundary data $g_1(\varphi)$ and $g_\pi(\rho)$.

Let us first consider the expansions obtained for $\psi_1(\rho,\varphi)$ and $\psi_\pi(\rho,\varphi)$ in terms of the eigenfunctions $\sin n\varphi$ (derived with the radial division of the problem domain), given by (26) and (27). These equations furnish

$$\psi_1^{\frown}(1,\varphi) = \sum_{n=1}^\infty \gamma_{1n}\sin n\varphi = g_1(\varphi) \ ,$$

since, in accordance with (28), γ_{1n} are the coefficients of the Fourier sine series for $g_1(\varphi)$, as well as

$$\psi_\pi^{\frown}(\rho,\pi) = \sum_{n=1}^\infty (-1)^n I_{\pi n}(\rho)\underbrace{\sin n\pi}_{0} = 0 \ .$$

Therefore, the expansion in the eigenfunctions $\sin n\varphi$ reproduces the boundary data $g_1(\varphi)$, but it fails do yield $g_\pi(\rho)$.

Consider now (29) and (30), the expansions of the boundary terms in terms of the continuous-spectrum eigenfunctions $\sin(k\ln\rho)$ (derived with the sectorial division of the problem domain); they furnish

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$$\psi_1^{\vee}(1,\varphi) = \int_0^{\infty} dk\, I_{1k}(\varphi) \frac{\sin(k\ln 1)}{\sinh k\pi} = 0$$

and

$$\psi_\pi^{\vee}(\rho,\pi) = \int_0^{\infty} dk\, \gamma_\pi(k) \sin ku = \bar{g}_\pi(u) = g_\pi(\rho) \ .$$

This last result is explained by the fact that, in view of (31), $\bar{g}_\pi(u)$ is the inverse Fourier sine transform of $\gamma_\pi(k)$.

Therefore, the expansion in the continuous-spectrum eigenfunctions $\sin(k\ln\rho)$ reproduces the boundary data $g_\pi(\rho)$, but if fails to yield $g_1(\varphi)$.

We thus conclude the following:

If the boundary data is a function of some variable, the corresponding Green's function boundary term "is better" expanded in terms of the eigenfunctions which are functions of that variable.

We say "is better" rather than "has to be" for the following reason.

Since both expressions $\psi^{\vee}$ and $\psi^{\frown}$ for the solution $\psi = \psi_f + \psi_1 + \psi_\pi$ of the problem converge everywhere in the (open) domain $\mathcal{A}$ (cf. Reference [7]), and, on the boundary $\partial\mathcal{A}$, ψ is known, we could consider unimportant the fact that $\psi_1^{\vee}$ and $\psi_\pi^{\frown}$ do not reproduce the boundary data $g_1(\varphi)$ and $g_\pi(\rho)$, respectively; but we can not! Indeed, it is a corollary of this fact that the convergence of $\psi_1^{\vee}$ and $\psi_\pi^{\frown}$ in $\mathcal{A}$ will be more difficult to achieve than that of $\psi_1^{\frown}$ and $\psi_\pi^{\vee}$, respectively.

b) Green's function $G^{\frown}(\rho,\varphi\,|\,\rho',\varphi)$ can be expressed in closed form. In fact, with the definitions $p \equiv \rho_< \rho_> = \rho'\rho$, $q \equiv \rho_< / \rho_>$, $d \equiv \varphi' - \varphi$ and $s \equiv \varphi' + \varphi$, we can develop (21) as follows:

$$\begin{aligned} G^{\frown}(\rho,\varphi\,|\,\rho',\varphi) &= \\ &= \sum_{n=1}^{\infty} \frac{-4}{n}(p^n - q^n)\frac{\cos nd - \cos ns}{2} \\ &= -2\sum_{n=1}^{\infty}\frac{p^n}{n}\cos nd + 2\sum_{n=1}^{\infty}\frac{q^n}{n}\cos nd \\ &\quad + 2\sum_{n=1}^{\infty}\frac{p^n}{n}\cos ns - 2\sum_{n=1}^{\infty}\frac{q^n}{n}\cos ns \ . \end{aligned} \tag{32}$$

However, notice that (cf. Eq. (5.2.31) in Reference [8])

$$\begin{aligned} -2\sum_{n=1}^{\infty}\frac{r^n}{n}\cos n\theta &= -2\,\mathrm{Re}\sum_{n=1}^{\infty}\frac{z^n}{n} \\ &= 2\,\mathrm{Re}\{\log(1-z)\} = 2\ln|1-z| \\ &= \ln(1 - 2r\cos\theta + r^2) \ , \end{aligned} \tag{33}$$

where $z = r\,e^{\mathrm{i}\theta}$, and the well-known Taylor's series of $\log(1-z)$ was used (in the above, we distinguish between the complex logarithmic function and the real one by employing the notations log and ln, respectively). Therefore, we can use the formula in (33) to replace each series in (32) by a logarithmic term, thus accomplishing our intent of expressing Green's function in a closed form:

$$\begin{aligned} G^{\frown}(\rho,\varphi\,|\,\rho',\varphi) &= \\ \ln(1-2p\cos d + p^2) &- \ln(1-2q\cos d + q^2) \\ -\ln(1-2p\cos s + p^2) &+ \ln(1-2q\cos s + q^2) \ . \end{aligned}$$

c) The integral which furnishes the Green's function $\bar{G}^{\vee}(u,\varphi\,|\,u',\varphi')$, in (25), can be evaluated by considering it along the closed contour of the *k*-plane shown in

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Figure 3 (where the radius tends to infinite: $R \to \infty$). This is possible because the integrand is an even function of k, allowing switching to half the integral from $-\infty$ to ∞. Then, since the poles of the integrand are the zeros of $k \sinh k\pi$, except the removable singularity $k=0$, that is, $m\pi \mathrm{i}$ $(m = \pm 1, \pm 2, \pm 3, ...)$, it is a simple matter to show, by using the residue theorem, that

$$\bar{G}^{\vee}(u,\varphi \mid u',\varphi') = \frac{1}{2} 2\pi \mathrm{i} \sum_{n=1}^{\infty} \mathrm{Res}(m\mathrm{i}) =$$

$$\sum_{m=1}^{\infty} \frac{8(-1)^m}{m} \sinh mu \sinh mu' \sin m\varphi_< \sin m(\pi - \varphi_>) \ .$$

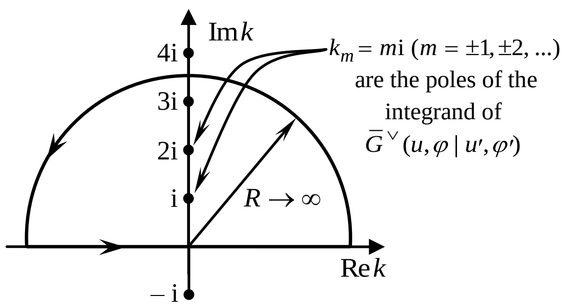

Figure 3 – The closed contour used to evaluate the real integral in (25) with the help of the residue theorem.

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