March, 2014. Vol. 4, No. 10 ISSN2305-8269
International Journal of Engineering and Applied Sciences

www.eaas-journal.org

SOME CHARACTERIZATIONS FOR GEODESIC SPRAYS IN DUAL SPACE

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ABSTRACT

In this study, firstly, the natural lift $\bar{\alpha}$ of a curve α and geodesic spray concepts are defined in dual space. Then, "The natural lift $\bar{\alpha}$ of a curve α is an integral curve of the geodesic spray X if and only if α is a geodesic on M'' which was given by Thorpe has been proved in dual space and it has been done applications for the natural lift curves of the spherical indicatries of tangent, principal normal, binormal vectors and the fixed centrode of a curve. Furthermore, some interesting results are obtained depending on the assumption the natural lift curves should be the integral curve of the geodesic spray on the bundle in dual space.

Keywords: Dual Frenet frames, geodesic spray, lift curve.

I.INTRODUCTION AND PRELIMINARIES

Let α be a curve with curvature κ and torsion τ.We denote by {T(t), N(t), B(t)} the moving Frenet frame along the curve α. Then, Frenet formulas are given by

$$\vec{T} = \vec{t} + \varepsilon\vec{t}^*,$$

$$\vec{N} = \vec{n} + \varepsilon\vec{n}^*,$$

$$\vec{B} = \vec{b} + \varepsilon\vec{b}^*.$$

Curvature κ and torsion τ

$$\kappa = k_1 + \varepsilon k_1^*,$$

$$\tau = k_2 + \varepsilon k_2^*.$$

We can write

$$\vec{t}' = k_1\vec{n},$$

$$\vec{n}' = -k_1\vec{t} + k_2\vec{b},$$

$$\vec{b}' = -k_2\vec{n},$$

$$t^{*'} = k_1 n^*,$$

$$n^{*'} = b - k_1 t^* + k_2 b^*,$$

$$b^{*'} = -n - k_2 n^*.$$

II.THE NATURAL LIFT OF CURVES AND GEODESIC SPRAY

Definition II.1 Let M be a hypersurface in dual space and let α: I→ M be a parametrized curve. α is called an integral curve of X if

$$\frac{d}{dt}(\alpha(t)) = X(\alpha(t)) \qquad \text{(for all } t\in I) \tag{1}$$

where X is a smooth tangent vector field on M. We have

$$TM = \bigcup T_p M, \; p\in M \tag{2}$$

where T_pM is the tangent space of M at p and χ(M) is the space of vector fields of M.

Definition II.2 $\bar{\alpha}$:I→TM given by

$$\bar{\alpha}(t) = (\alpha(t), \dot{\alpha}(t)) = \dot{\alpha}(t)|_{\alpha(t)} \tag{3}$$

is called the natural lift of α on TM.Thus,we can write

$$\frac{d\bar{\alpha}}{dt} = \frac{d}{dt}(\dot{\alpha}(t)|_{\alpha(t)}) = D_{\dot{\alpha}(t)}\dot{\alpha}(t) \tag{4}$$

where D is the Levi-Civita connection in dual space.

ISSN2305-8269
International Journal of Engineering and Applied Sciences

www.eaas-journal.org

Definition II.3 For v∈ TM, the smooth vector field X∈ χ(M) defined by

$$X (v) =-<v, S (v) >N|_{p} \tag{5}$$

is called the geodesic spray on the manifold TM, where N is the unit normal vector field of M.

Theorem II.1: Let α be a curve on the unit dual sphere. The natural lift $\bar{\alpha}$ of the curve α is an integral curve of the geodesic spray X if and only if α is a geodesic on the unit dual sphere.

Proof. Let α(t) = α_1(t)+$\varepsilon\alpha^*{}_1$(t) be an integral curve of the geodesic spray X. Thus

$$\frac{d}{dt}(\alpha(t))=X (\alpha(t)) = (\alpha_1(t) +\varepsilon\alpha^*{}_1(t))|_{\alpha(t)}. \tag{6}$$

Since X is a geodesic spray on the unit dual sphere, we have

$$X(\bar{\alpha}(t))=-< \bar{\alpha}(t),S(\bar{\alpha}(t))>N|_{\alpha(t)}. \tag{7}$$

Form (6) and (7) we get

$$\frac{d}{dt}(\dot{\alpha}(t) |_{\alpha(t)}=-< \dot{\alpha}(t),S(\dot{\alpha}(t))>N|_{\alpha(t)}. \tag{8}$$

Since last equation is true for all α(t) using (4) and S=I_2

$$D_{\dot{\alpha(t)}}\dot{\alpha}(t) =-< \dot{\alpha}(t), S (\dot{\alpha}(t)) >N|_{\alpha(t)} =0. \tag{9}$$

Thus from the last equation and Gauss Equation we have

$$\bar{D}_{\dot{\alpha(t)}}\dot{\alpha}(t)=0, \tag{10}$$

where $\bar{D}$ is the Gauss-Connection on the unit dual sphere. Hence, we have seen that α is a geodesic on the unit dual sphere.

Now, assume that α be a geodesic on the unit dual sphere. Then

$$\bar{D}_{\dot{\alpha(t)}}\dot{\alpha}(t)=0.$$

Hence, by the Gauss-Equation we have

$$D_{\dot{\alpha(t)}}\dot{\alpha}(t)|_{\alpha(t)} +< \dot{\alpha}(t), S (\dot{\alpha}(t))>N|_{\alpha(t)} =0.$$

Since X is the geodesic spray, we can write

$$\frac{d}{dt}(\dot{\alpha}(t))|_{\alpha(t)} -X(\dot{\alpha}(t)) |_{\alpha(t)}=0,$$

$$\frac{d}{dt}(\dot{\alpha}(t))|_{\alpha(t)} =X(\dot{\alpha}(t)) |_{\alpha(t)},$$

$$\frac{d}{dt}(\dot{\alpha}(t)) =X(\dot{\alpha}(t)).$$

II.1.The Natural Lift Of The Spherical Indicatrix Of Tangent Vectors Of α

We will investigate how α must be a curve satisfying the condition that $\bar{\alpha}_T$ is an integral curve of the geodesic spray, where α_T being the spherical indicatrix of tangent vectors of α, $\bar{\alpha}_T$ is the natural lift of the curve α_T.

Let α be a curve on the unit dual sphere. If $\bar{\alpha}_T$ is an integral curve of the geodesic spray, then by means of Theorem II.1

$$\bar{D}_{\dot{\alpha}_T}\dot{\alpha}_T=0,$$

that is,

$$D_{\dot{\alpha}_T}\dot{\alpha}_T+< \dot{\alpha}_T, S (\dot{\alpha}_T)> T(s)=0,$$

where s is the arc-length of α.

Since S=I_2 for the unit sphere, we have

$$D_{\dot{\alpha}_T}\dot{\alpha}_T+\|\dot{\alpha}_T\|^2T(s) = 0,$$

$$D_{\dot{\alpha}_T}((k_1+\varepsilon k_1^*) (n+\varepsilon n^*)) + \|\dot{\alpha}_T\|^2(t+\varepsilon t^*)=0,$$

$$\frac{d}{ds_T}((k_1+\varepsilon k_1^*) (n+\varepsilon n^*)) + \|\dot{\alpha}_T\|^2(t+\varepsilon t^*)=0,$$

where s_T is the arc-length of α_T. After some algebraic calculation we find that

March, 2014. Vol. 4, No. 10 ISSN2305-8269
International Journal of Engineering and Applied Sciences

www.eaas-journal.org

t$(-k_1+k^2{}_1-\varepsilon k_1{}^*)$+n$(\frac{\dot{k}_1}{k_1}+\varepsilon\frac{\dot{k}_{1*}}{k_1})$+b$(k_2+\varepsilon(\frac{k_{1*}}{k_1}k_2+1))$+$\varepsilon(\frac{\dot{k}_1}{k_1}$n*+t*$(-k_1+k^2{}_1)+k_2$b*)=0.

Since T, N, B are linear independent, we have

$-k_1+k^2{}_1-\varepsilon k_1{}^*=0,$

$\frac{\dot{k}_1}{k_1}+\varepsilon\frac{\dot{k}_{1*}}{k_1}=0,$

$k_2+\varepsilon\left(\frac{k_{1*}}{k_1}k_2+1\right)=0,$

$-k_1+k^2{}_1=0,$

k_2=0.

Corollary I There is no curve α whose the spherical indicatrix α_T is a great circle on the unit dual sphere.Therefore,the natural lift $\bar{\alpha}_T$ of the curve α_T can never be an integral curve of the geodesic spray on the tangent bundle T(S^2).

II.2.The Natural Lift Of The Spherical Indicatrix Of Principal Normal Vectors Of α

We will investigate how α must be a curve satisfying the condition that $\bar{\alpha}_N$ is an integral curve of the geodesic spray,where α_N being the spherical indicatrix of principal normal vectors of α, $\bar{\alpha}_N$ is the natural lift of the curve α_N.

Let α be a curve on the unit dual sphere. If $\bar{\alpha}_N$ is an integral curve of the geodesic spray, then by means of Theorem II.1

$$\bar{D}_{\dot{\alpha}_N}\dot{\alpha}_N=0,$$

that is,

$$D_{\dot{\alpha}_N}\dot{\alpha}_N+<\dot{\alpha}_N, S(\dot{\alpha}_N)>N(s)=0,$$

where s is the arc-length of α .

Since S=I_2 for the unit sphere, we have

$$D_{\dot{\alpha}_N}\dot{\alpha}_N+\|\dot{\alpha}_N\|^2N(s)=0,$$

$D_{\dot{\alpha}_N}(-(k_1+\varepsilon k_1{}^*)$ (t+εt*) $+(k_2+\varepsilon k_2{}^*)$(b+εb*)) + $\|\dot{\alpha}_N\|^2$(n+εn*)=0,

$\frac{d}{d s_N}(-(k_1+\varepsilon k_1{}^*)$ (t+εt*)+$(k_2+\varepsilon k_2{}^*)$(b+εb*)) + $\|\dot{\alpha}_N\|^2$(n+εn*)=0,

where s_N is the arc-length of α_N. After some algebraic calculation we find that

t$(-\dot{k}_1-\varepsilon\dot{k}_1{}^*)$+n$(\|w\|^3-\|w\|^2-k_2+\varepsilon(k_1{}^*k_1+k_2{}^*k_2))$+b$(\dot{k}_2+\varepsilon\dot{k}_2{}^*)$+t*$(-\varepsilon\dot{k}_1)$+n*$(-k^2{}_2+\varepsilon(\|w\|^3-k^2{}_1))+\dot{k}_2$b*=0,

where w is the Darboux vector.

Since T, N, B are linear independent, we have

$-\dot{k}_1-\varepsilon\dot{k}_1{}^*=0.$ ($k_1{}^*$=const.) ;

$\|w\|^3-\|w\|^2-k_2=0,$

$k_1{}^*k_1+k_2{}^*k_2=0,$

$\dot{k}_2+\varepsilon\dot{k}_2{}^*=0.$ (k_2=const., $k_2{}^*$=const.) ;

$\dot{k}_1=0,$

$k^2{}_2=0,$

$\|w\|^3-k^2{}_1=0,$

$\dot{k}_2=0,$ (k_2=const.).

Corollary II If the curve α is a circular helix on the unit dual sphere, then its spherical indicatrix α_N is a great circle on the unit sphere. In this case, the natural lift $\bar{\alpha}_N$ of α_N is an integral curve of the geodesic spray on the tangent bundle T (S^2).

II.3.The Natural Lift Of The Spherical Indicatrix Of Binormal Vectors Of α

We will investigate how α must be a curve satisfying the condition that $\bar{\alpha}_B$ is an integral curve of the geodesic spray,where α_B being the spherical indicatrix of binormal vectors of α, $\bar{\alpha}_B$ is the natural lift of the curve α_B.

Let α be a curve on the unit dual sphere.If $\bar{\alpha}_B$ is an integral curve of the geodesic spray, then by means of Theorem II.1

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$$\bar{D}_{\dot{\alpha}_B}\dot{\alpha}_B=0,$$

that is,

$$D_{\dot{\alpha}_B}\dot{\alpha}_B+<\dot{\alpha}_B, S(\dot{\alpha}_B)> B(s)=0$$

where s is the arc-length of α.

Since S=I_2 for the unit sphere, we have

$$D_{\dot{\alpha}_B}\dot{\alpha}_B+\|\dot{\alpha}_B\|^2 B(s) = 0,$$

$$D_{\dot{\alpha}_B}(-(k_2+\varepsilon k_2*)(n+\varepsilon n*)+k_2{}^2(b+\varepsilon b*))=0,$$

$$\frac{d}{d s_B}(-k_2 n-\varepsilon(k_2 n*+k_2*n))+k_2{}^2(b+\varepsilon b*)=0$$

where s_N is the arc-length of α_N.After some algebraic calculation we find that

$$t\,(k_1-\varepsilon\frac{k_1 k_2*}{k_2})+n(-\frac{\dot{k}_2}{k_2}-\varepsilon\dot{k}_2*)+b\,(k_2{}^2-k_2+\varepsilon(k_2*-1))+t*(k_1\varepsilon)+n*(-\frac{\varepsilon\dot{k}_2}{k_2})+b*\varepsilon(k_2{}^2-k_2)=0\,.$$

Since T, N, B are linear independent, we have

$$k_1-\varepsilon\frac{k_1 k_2*}{k_2}=0,$$

$$-\frac{\dot{k}_2}{k_2}-\varepsilon\dot{k}_2*=0,$$

$$k_2{}^2-k_2+\varepsilon(k_2*-1)=0,$$

$$k_1=0,$$

$$\frac{\varepsilon\dot{k}_2}{k_2}=0.$$

Corollary III There is no special position of the curve. The natural lift $\bar{\alpha}$ of the curve α is an integral curve of the geodesic spray X if and only if α is a geodesic on the unit dual sphere.

II.4.On The Natural Lift Of The Fixed Centrode

Let α_C be the fixed centrode of the motion by the curve α. Then the curve is given by

$$\alpha_C= C(s)$$

and

$$C=\frac{w}{\|w\|},$$

where w being the Darboux vector.

If ϕ=ϕ(s) denotes the angle between B and C, then we have

$$k_1=\|w\|\cos\phi,$$

$$k_2=\|w\|\sin\phi.$$

We will investigate how α must be a curve satisfying the condition that $\bar{\alpha}_C$ is an integral curve of the geodesic spray. Let the natural lift $\bar{\alpha}_C$ of α_C be an integral curve of the geodesic spray. According to the Theorem II.1, we write

$$\bar{D}_{\dot{\alpha}_C}\dot{\alpha}_C=0,$$

$$D_{\dot{\alpha}_C}\dot{\alpha}_C+<\dot{\alpha}_C, S(\dot{\alpha}_C)> C(s)=0$$

where s is the arc-length of α.

Since S=I_2 for the unit sphere, we have

$$\frac{d}{d s_C}(\dot{\alpha}_C)+\|\dot{\alpha}_C\|^2 C=0,$$

where s_C is the arc-length of the curve α_C and C=sin(ϕ+εϕ*)T+ cos (ϕ+εϕ*)B.

After some algebraic calculation we find that

$$T\,(-\sin(\phi+\varepsilon\phi*)(\phi+\varepsilon\phi*)^2+(\ddot{\phi}+\varepsilon\ddot{\phi}*)\cos(\phi+\varepsilon\phi*)+\sin(\phi+\varepsilon\phi*)(\phi+\varepsilon\phi*)^3)+$$

$$N(k_1(\dot{\phi}+\varepsilon\dot{\phi}*)\cos(\phi+\varepsilon\phi*)+k_2(\dot{\phi}+\varepsilon\dot{\phi}*)\sin(\phi+\varepsilon\phi*))+$$

$$B\,(-\cos(\phi+\varepsilon\phi*)(\dot{\phi}+\varepsilon\dot{\phi}*)^2-\sin(\phi+\varepsilon\phi*)(\ddot{\phi}+\varepsilon\ddot{\phi}*)+(\dot{\phi}+\varepsilon\dot{\phi}*)^3\cos(\dot{\phi}+\varepsilon\dot{\phi}*))=0$$

The last equation imply that $\dot{\phi}+\varepsilon\dot{\phi}*=0$ or $k_1=k_2=0$. Since $\dot{\phi}+\varepsilon\dot{\phi}*=0$, $\frac{k_1}{k_2}$ is constant.

This implies that α is a dual helix.The condition $k_1=k_2=0$ implies that α is a line. In the second case, we don't have a solution. Then we have the following result.

Corollary IV If the curve α is a dual helix, then, its fixed centrode α_C is a great circle on the unit dual sphere. In this case, the natural lift $\bar{\alpha}_C$ of α_C is an integral curve of the geodesic spray on the tangent bundle T (S^2).

ISSN2305-8269
International Journal of Engineering and Applied Sciences

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