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ON GENERALIZATION OF SOME INTEGRAL INEQUALITIES FOR QUASI-CONVEX FUNCTIONS AND THEIR APPLICATONS

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ABSTRACT

In this paper, we establish some new integral inequalities for quasi-convex functions. From our results the classical the midpoint, trapezoid, and Simpson inequalities can be deduced as some special cases. Some applications to special means of real numbers are also given

KEYWORDS: Quasi-convex function, Simpson's inequality, Hermite-Hadamard's inequality, midpoint inequality, trapezoid ineqaulity.

1. INTRODUCTION

Let $f: I \subset R \to R$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \quad (1.1)$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions.Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if f is concave. See [1,3,4,6,7,9,10], the results of the generalization, improvement and extention of the famous integral inequality (1.1).

The notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function $f:[a,b] \to R$ is said quasi-convex on $[a,b]$ if

$$f(\alpha x + (1-\alpha)y) \leq \sup\{f(x), f(y)\},$$

for any $x, y \in [a,b]$ and $\alpha \in [0,1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see [7]).

The following inequality is well known in the literature as Simpson's inequality .

Let $f:[a,b] \to R$ be a four times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_\infty = \sup_{x\in(a,b)}\left|f^{(4)}(x)\right| < \infty$. Then the following inequality holds:

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^2.$$

In recent years many authors have studied error estimations for Simpson's inequality; for refinements, counterparts, generalizations and new Simpson's type inequalities, see [2,5,11,12,13]

In [3], Alomari et al. established some upper bound for the right -hand side of Hadamard's inequality

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for quasi-convex mappings, The authors obtained the following results:

Theorem 1.1. Let $f: I \subset R \to R$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $\left|f'\right|^{p/(p-1)}$ is an quasi-convex on $[a,b]$, for $p>1$, then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{4(p+1)^{1/p}}\left[\left(\sup\left\{\left|f'(\frac{a+b}{2})\right|^{\frac{p}{p-1}},\left|f'(b)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right.$$

$$\left.+\left(\sup\left\{\left|f'(\frac{a+b}{2})\right|^{\frac{p}{p-1}},\left|f'(a)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right]. \qquad (1.2)$$

Theorem 1.2. Let $f: I^{\circ} \subset R \to R$ be a differentiable mapping on I°, $a,b \in I^{\circ}$ with $a<b$. If $\left|f'\right|^{q}$ is an quasi-convex on $[a,b]$, for $q \geq 1$, then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{8}\left[\left(\sup\left\{\left|f'(\frac{a+b}{2})\right|^{q},\left|f'(b)\right|^{q}\right\}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\sup\left\{\left|f'(\frac{a+b}{2})\right|^{q},\left|f'(a)\right|^{q}\right\}\right)^{\frac{1}{q}}\right]. \qquad (1.3)$$

In this paper, in order to provide a unified approach to establish midpoint inequality, trapezoid inequality and Simpson's inequality for functions whose derivatives in absolute value at certain power are quasi-convex, we need the following lemma:

Lemma 1.3. Let $f: I \subset R \to R$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$ and $\theta,\lambda \in [0,1]$, then the following equality holds:

$$(1-\theta)\left(\lambda f(a)+(1-\lambda)f(b)\right)+\theta f((1-\lambda)a+\lambda b)$$

$$-\frac{1}{b-a}\int_a^b f(x)dx$$

$$=(b-a)\left[-\lambda^2\int_0^1 (t-\theta)f'\left(ta+(1-t)\left[(1-\lambda)a+\lambda b\right]\right)dt\right.$$

$$\left.+(1-\lambda)^2\int_0^1 (t-\theta)f'\left(tb+(1-t)\left[(1-\lambda)a+\lambda b\right]\right)dt\right].$$

(1.4)

A simple proof of equality can be given by performing an integration by parts in the integrals from the right side and changing the variable (see [8]).

2. MAIN RESULTS

Theorem 2.1. Let $f: I \subset R \to R$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a,b \in I^{\circ}$ with $a<b$ and $\alpha,\lambda \in [0,1]$. If $\left|f'\right|^{q}$ is quasi-convex on $[a,b]$, $q \geq 1$, then the following inequality holds:

$$\left|(1-\theta)\left(\lambda f(a)+(1-\lambda)f(b)\right)+\theta f((1-\lambda)a+\lambda b)\right.$$

$$\left.-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq (b-a)\left(\theta^2-\theta+\frac{1}{2}\right)\left[\lambda^2\left(\sup\left\{\left|f'(a)\right|^{q},\left|f'(C)\right|^{q}\right\}\right)^{\frac{1}{q}}\right.$$

$$\left.+(1-\lambda)^2\left(\sup\left\{\left|f'(b)\right|^{q},\left|f'(C)\right|^{q}\right\}\right)^{\frac{1}{q}}\right] \qquad (2.1)$$

where $C=(1-\lambda)a+\lambda b$.

Proof. Suppose that $q \geq 1$ and $C=(1-\lambda)a+\lambda b$. From Lemma 1.3 and using the well known power mean inequality, we have

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$$\begin{aligned}&\left|(1-\theta)\left(\lambda f(a)+(1-\lambda)f(b)\right)+\theta f(C)\right.\\&\left.-\frac{1}{b-a}\int_a^b f(x)dx\right|\\&\leq(b-a)\left[\lambda^2\int_0^1|t-\theta|\left|f'\left(ta+(1-t)C\right)\right|dt\right.\\&\left.+(1-\lambda)^2\int_0^1|t-\theta|\left|f'\left(tb+(1-t)C\right)\right|dt\right]\\&\leq(b-a)\left\{\lambda^2\left(\int_0^1|t-\theta|dt\right)^{1-\frac{1}{q}}\right.\\&\times\left(\int_0^1|t-\theta|\left|f'\left(ta+(1-t)C\right)\right|^q dt\right)^{\frac{1}{q}}\\&\left.+(1-\lambda)^2\left(\int_0^1|t-\theta|dt\right)^{1-\frac{1}{q}}\left(\int_0^1|t-\theta|\left|f'\left(tb+(1-t)C\right)\right|^q dt\right)^{\frac{1}{q}}\right\}.\end{aligned}\tag{2.2}$$

Since $\left|f'\right|^q$ is quasi-convex on $[a,b]$, we know that for $t\in[0,1]$

$$\left|f'\left(ta+(1-t)C\right)\right|^q\leq\sup\left\{\left|f'(a)\right|^q,\left|f'(C)\right|^q\right\},$$

and

$$\left|f'\left(tb+(1-t)C\right)\right|^q\leq\sup\left\{\left|f'(a)\right|^q,\left|f'(C)\right|^q\right\}.$$

Hence, by simple computation

$$\int_0^1|t-\theta|dt=\theta^2-\theta+\frac{1}{2},\tag{2.3}$$

$$\begin{aligned}&\int_0^1|t-\theta|\left|f'\left(ta+(1-t)C\right)\right|^q dt\\&=\left(\theta^2-\theta+\frac{1}{2}\right)\sup\left\{\left|f'(a)\right|^q,\left|f'(C)\right|^q\right\},\end{aligned}\tag{2.4}$$

and

$$\begin{aligned}&\int_0^1|t-\theta|\left|f'\left(tb+(1-t)C\right)\right|^q dt\\&=\left(\theta^2-\theta+\frac{1}{2}\right)\sup\left\{\left|f'(b)\right|^q,\left|f'(C)\right|^q\right\}.\end{aligned}\tag{2.5}$$

Thus, using (2.3)-(2.5) in (2.2), we obtain the inequality (2.1). This completes the proof. □

Corollary 2.2. Under the assumptions of Theorem 2.1 with $q=1$, the inequality (2.1) reduced to the following inequality

$$\begin{aligned}&\left|(1-\theta)\left(\lambda f(a)+(1-\lambda)f(b)\right)+\theta f((1-\lambda)a+\lambda b)\right.\\&\left.-\frac{1}{b-a}\int_a^b f(x)dx\right|\\&\leq(b-a)\left(\theta^2-\theta+\frac{1}{2}\right)\left[\lambda^2\sup\left\{\left|f'(a)\right|,\left|f'(C)\right|\right\}\right.\\&\left.+(1-\lambda)^2\sup\left\{\left|f'(b)\right|,\left|f'(C)\right|\right\}\right].\end{aligned}$$

Corollary 2.3. Under the assumptions of Theorem 2.1 with $\lambda=\frac{1}{2}$ and $\theta=\frac{2}{3}$, from the inequality (2.1) we get the following Simpson type inequality

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\\&\leq(b-a)\left(\frac{5}{72}\right)\left[\left(\sup\left\{\left|f'(a)\right|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right.\\&\left.+\left(\sup\left\{\left|f'(b)\right|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right].\end{aligned}$$

Corollary 2.4. Under the assumptions of Theorem 2.1 with $\lambda=\frac{1}{2}$ and $\theta=1$, from the inequality (2.1) we get the following midpoint inequality

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$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq\frac{b-a}{8}\left[\left(\sup\left\{\left|f'(a)\right|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\sup\left\{\left|f'(b)\right|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right].$$

Corollary 2.5. Under the assumptions of Theorem 2.1 with $\lambda=\frac{1}{2}$ and $\theta=0$, from the inequality (2.1) we get the following trapezoid inequality

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq\frac{b-a}{8}\left[\left(\sup\left\{\left|f'(a)\right|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\sup\left\{\left|f'(b)\right|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right].$$

which is the same of the inequality (1.3).
Using Lemma 1.3 we shall give another result for convex functions as follows.

Theorem 2.6. Let $f:I\subset R\to R$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I^\circ$ with $a<b$ and $\alpha,\lambda\in[0,1]$. If $\left|f'\right|^q$ is quasi-convex on $[a,b]$, $q>1$, then the following inequality holds:

$$\left|(1-\theta)\left(\lambda f(a)+(1-\lambda)f(b)\right)+\theta f((1-\lambda)a+\lambda b)\right.$$
$$\left.-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq(b-a)\left(\frac{\theta^{p+1}+(1-\theta)^{p+1}}{p+1}\right)^{\frac{1}{p}}$$
$$\times\left[\lambda^2\left(\sup\left\{\left|f'(a)\right|^q,\left|f'(C)\right|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+(1-\lambda)^2\left(\sup\left\{\left|f'(b)\right|^q,\left|f'(C)\right|^q\right\}\right)^{\frac{1}{q}}\right] \quad (2.6)$$

where $C=(1-\lambda)a+\lambda b$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Suppose that $C=(1-\lambda)a+\lambda b$. From Lemma 1.3 and by Hölder's integral inequality, we have

$$\left|(1-\theta)\left(\lambda f(a)+(1-\lambda)f(b)\right)+\theta f(C)\right.$$
$$\left.-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq(b-a)\left[\lambda^2\int_0^1|t-\theta|\left|f'\left(ta+(1-t)C\right)\right|dt\right.$$
$$\left.+(1-\lambda)\int_0^1|t-\theta|\left|f'\left(tb+(1-t)C\right)\right|dt\right]$$

$$\leq(b-a)\left\{\lambda^2\left(\int_0^1|t-\theta|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(ta+(1-t)C\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+(1-\lambda)^2\left(\int_0^1|t-\theta|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(tb+(1-t)C\right)\right|^q dt\right)^{\frac{1}{q}}\right\}.$$
(2.7)

Since $\left|f'\right|^q$ is quasi-convex on $[a,b]$, we get

$$\int_0^1\left|f'\left(ta+(1-t)C\right)\right|^q dt=\sup\left\{\left|f'(a)\right|^q,\left|f'(C)\right|^q\right\}$$
(2.8)

Similarly,

$$\int_0^1\left|f'\left(tb+(1-t)C\right)\right|^q dt=\sup\left\{\left|f'(b)\right|^q,\left|f'(C)\right|^q\right\}.$$
(2.9)

By simple computation

$$\int_0^1|t-\theta|^p dt=\frac{\theta^{p+1}+(1-\theta)^{p+1}}{p+1}, \quad (2.10)$$

thus, using (2.8)-(2.10) in (2.7), we obtain the inequality (2.6). This completes the proof. □

Corollary 2.7. Under the assumptions of Theorem 2.6 with $\lambda=\frac{1}{2}$ and $\theta=\frac{2}{3}$, from the inequality (2.6) we get the following Simpson type inequality

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$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq\frac{b-a}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left\{\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\right\}.$$

Corollary 2.8. Under the assumptions of Theorem 2.6 with $\lambda=\frac{1}{2}$ and $\theta=0$, from the inequality (2.6) we get the following trapezoid inequality

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq\frac{b-a}{8}\left[\left(\sup\left\{\left|f'(a)\right|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\sup\left\{\left|f'(b)\right|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right].$$

which is the same of the inequality (1.2)

Corollary 2.9. Under the assumptions of Theorem 1 with $\lambda=\frac{1}{2}$ and $\theta=1$, from the inequality (1) we get the following midpoint inequality

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq\frac{b-a}{4(p+1)^{1/p}}\left[\left(\sup\left\{\left|f'(a)\right|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\sup\left\{\left|f'(b)\right|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right].$$

which is the better than the inequality in [1, Corollary 8].

3. SOME APPLICATIONS FOR SPECIAL MEANS

Let us recall the following special means of arbitrary real numbers a,b with $a\neq b$ and $\alpha\in[0,1]$:

1. The weighted arithmetic mean
$$A_\alpha(a,b):=\alpha a+(1-\alpha)b,\ a,b\in R.$$

2. The unweighted arithmetic mean
$$A(a,b):=\frac{a+b}{2},\ a,b\in R.$$

3. The weighted harmonic mean
$$H_\alpha(a,b):=\left(\frac{\alpha}{a}+\frac{1-\alpha}{b}\right)^{-1},\ a,b\in R\setminus\{0\}.$$

4. The unweighted harmonic mean
$$H(a,b):=\frac{2ab}{a+b},\ a,b\in R\setminus\{0\}.$$

5. The Logarithmic mean
$$L(a,b):=\frac{b-a}{\ln b-\ln a},\ a,b>0,a\neq b.$$

6. Then n-Logarithmic mean
$$L_n(a,b):=\left(\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right)^{\frac{1}{n}},n\in N,a,b\in R,a\neq b.$$

Proposition 3.1. Let $a,b\in R$ with $a<b$, and $n\in N,n\geq 2.$ Then, for $\theta,\lambda\in[0,1]$ and $q\geq 1$, we have the following inequality:

$$\left|(1-\theta)A_\lambda\left(a^n,b^n\right)+\theta A_\lambda^n(a,b)-L_n^n(a,b)\right|$$
$$\leq(b-a)\left(\theta^2-\theta+\frac{1}{2}\right)n\left[\lambda^2\left(\sup\left\{|a|^{(n-1)q},\left|A_\lambda(b,a)\right|^{(n-1)q}\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+(1-\lambda)^2\left(\sup\left\{|b|^{(n-1)q},\left|A_\lambda(b,a)\right|^{(n-1)q}\right\}\right)^{\frac{1}{q}}\right].$$

Proof. The assertion follows from Theorem 2.1, for $f(x)=x^n,x\in R.$ □

Proposition 3.2. Let $a,b\in R$ with $a<b$, and $n\in N,n\geq 2.$ Then, for $\theta,\lambda\in[0,1]$ and $q>1$, we have the following inequality:

$$\left|(1-\theta)A_\lambda\left(a^n,b^n\right)+\theta A_\lambda^n(a,b)-L_n^n(a,b)\right|$$
$$\leq(b-a)\left(\frac{\theta^{p+1}+(1-\theta)^{p+1}}{p+1}\right)^{\frac{1}{p}}n\left[\lambda^2\left(\sup\left\{|a|^{(n-1)q},\left|A_\lambda(b,a)\right|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+(1-\lambda)^2\left(\sup\left\{|b|^{(n-1)q},\left|A_\lambda(b,a)\right|^q\right\}\right)^{\frac{1}{q}}\right]$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. The assertion follows from Theorem 2.6, for $f(x)=x^n, x\in R$. □

Proposition 3.3. Let $a,b\in R$ with $0<a<b$, and $\theta,\lambda\in[0,1]$. Then, for $q\geq 1$, we have the following inequality:

$$\left|(1-\theta)H_\lambda^{-1}(a,b)+\theta A_\lambda^{-1}(a,b)-L^{-1}(a,b)\right|$$
$$\leq(b-a)\left(\theta^2-\theta+\frac{1}{2}\right)\left[\lambda^2\left(\sup\left\{a^{-2q},A_\lambda(b,a)^{-2q}\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+(1-\lambda)^2\left(\sup\left\{b^{-2q},A_\lambda(b,a)^{-2q}\right\}\right)^{\frac{1}{q}}\right].$$

Proof. The assertion follows from Theorem 2.1., for $f(x)=\frac{1}{x}, x\in(0,\infty)$. □

Proposition 3.4. Let $a,b\in R$ with $0<a<b$, and $\theta,\lambda\in[0,1]$. Then, for $q>1$, we have the following inequality:

$$\left|(1-\theta)H_\lambda^{-1}(a,b)+\theta A_\lambda^{-1}(a,b)-L^{-1}(a,b)\right|$$
$$\leq(b-a)\left(\frac{\theta^{p+1}+(1-\theta)^{p+1}}{p+1}\right)^{\frac{1}{p}}\left[\lambda^2\left(\sup\left\{a^{-2q},A_\lambda(b,a)^{-2q}\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+(1-\lambda)^2\left(\sup\left\{b^{-2q},A_\lambda(b,a)^{-2q}\right\}\right)^{\frac{1}{q}}\right]$$

Proof. The assertion follows from Theorem 2.6, for $f(x)=\frac{1}{x},\ x\in(0,\infty)$. □

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