January 2013. Vol. 1, No.3 ISSN 2305-8269

International Journal of Engineering and Applied Sciences

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CONFIDENCE INTERVALS FOR THE QUANTILE OF THE QUADRATIC-NORMAL DISTRIBUTION

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ABSTRACT

Suppose we have a random sample from a non-normal distribution known as the quadratic-normal distribution. We construct 100(1-α) % confidence intervals for the γ-quantile of the quadratic-normal distribution using the procedures based on bootstrap, normal approximation and hypothesis testing. It is found that the coverage probability of the confidence interval based on the hypothesis testing tends to be closer to the target value than those of the bootstrap confidence interval and the confidence interval based on normal approximation.

KEYWORDS: *Confidence interval, Quantile, Hypothesis testing*

1. INTRODUCTION

Let $y_1, y_2, \ldots, y_n$ denote a random sample from a distribution with distribution function $F_{\boldsymbol{\eta}}$ indexed by the parameter vector $\boldsymbol{\eta}$. Assume that the γ-quantile, $Q_\gamma = F_{\boldsymbol{\eta}}^{-1}(\gamma)$, is uniquely defined.

For certain random variables, it is important to know the tails of their distributions. Examples of such random variables are the daily temperature, hourly or daily wind speed, sea level, wave height, pollutant concentration, magnitude of biological invasion impact, strength of glass fiber, and return of a financial investment. An important characteristic of the tails of distribution is the γ-quantile where γ is either very small or very large. It is important to quantify the accuracy of the estimated γ-quantile. A natural way to gauge the accuracy of the estimated γ-quantile is to construct a confidence interval for the theoretical value of the γ-quantile .

When the distribution of the y_i is normal, a two-sided 100(1- α)% confidence interval for Q_γ is proposed in Odeh and Owen(1980).

When the distribution of the y_i is not known, we may use the bootstrap procedure (see for example, Efron, 1982 and 1987).

The method of empirical likelihood was introduced by Owen (1988,1990) as an alternative to the bootstrap for constructing confidence regions in nonparametric problems. It has sampling properties similar to the bootstrap, but where the bootstrap uses resampling, it profiles the likelihood of a general multinomial distribution with one parameter per (distinct) data point. Its properties have been investigated in works by Owen (1990), Hall (1990) and DiCiccio, Hall and Romano (1991).

When applied to the problem of constructing confidence intervals for a population quantile, empirical likelihood essentially yields the

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sign-test interval which usually has a poor coverage probability.

Song & Hall (1993) showed that by appropriately smoothing the empirical likelihood method, coverage accuracy may be improved from order $n^{-1/2}$ to order n^{-1}.

One competing approach to determining confidence intervals for quantiles is based on interpolation of sign-test intervals. This method has only been developed in the case of the median (see, for example, Sheather, 1987 and Sheather and McKean, 1987).

The works on the construction of confidence interval for the γ-quantile can also be found in Kupiec (1995) and Pritsker (1996) where standard theory of order statistics has been used, in Jorion (1996) and Butler and Schachter (1996) where kernal theory is the major tool, and in Gomes and Pestana (2007) where the semi-parametric technique has been utilized.

In this paper, we assume that the random variable y_i has a type of non-normal distribution called the quadratic-normal distribution with parameters μ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)^T$ (see Pooi, A. H. (2003), Effects of non-normality on confidence intervals in linear models. Technical Report No. 6/2003. Institute of Mathematical Sciences, University of Malaya.).

We find confidence intervals for the γ-quantile using a hypothesis testing procedure which has been used in Ng and Pooi (2008), Goh, Y. L. (2008) entitled " Confidence Intervals for the quantiles of the quadratic-normal distribution" (PhD. Thesis, Institute of Mathematical Sciences, University of Malaya.) and Tan, S.K.(2009) entitled "Confidence intervals in survival analysis" (M.Sc. Thesis, Institute of Mathematical Sciences, University of Malaya.) to construct confidence intervals for individual parameters in the presence of nuisance parameters. It is found that the coverage probability of the confidence interval based on the hypothesis testing for the γ-quantile tends to be closer to the target value than those of the bootstrap confidence interval and the confidence interval based on normal approximation.

2. CONFIDENCE INTERVALS FOR γ-QUANTILE WHEN THE ERRORS HAVE QUADRATIC-NORMAL DISTRIBUTION

Let $y_i = \mu + \varepsilon_i$ and assume that the random variable ε_i can be expressed as

$$\varepsilon_i = \begin{cases} \lambda_1 e_i + \lambda_2 (e_i^2 - \frac{1+\lambda_3}{2}), & e_i \geq 0 \\ \lambda_1 e_i + \lambda_2 (\lambda_3 e_i^2 - \frac{1+\lambda_3}{2}), & e_i < 0 \end{cases}, i = 1,2,\cdots,n \quad (1)$$

where e_i ~ N(0,1), and $\lambda_1, \lambda_2, \lambda_3$ are parameters such that ε_i is a one-to-one function of e_i (see Pooi(2003)). The random variable ε_i is said to have a quadratic-normal distribution and we write $\varepsilon_i \sim QN(0, \boldsymbol{\lambda})$.

Let $m_k = E(\varepsilon_i^k)$, $k = 1,2,3,4$. Next let $\overline{m}_3 = m_3 / \{m_2\}^{\frac{3}{2}}$ and $\overline{m}_4 = m_4 / \{m_2\}^2$ be respectively the measures of skewness and kurtosis of the distribution of ε_i. The possible values of $(\overline{m}_3, \overline{m}_4)$ such that the function $\varepsilon_i = \varepsilon_i(e_i)$ given by Equation (1) is a one-to-one function may be represented by the region ***R*** in Figure (1).

With suitable choice of λ_1, λ_2 and λ_3, the quadratic-normal distribution can exhibit large skewness and kurtosis. Thus quadratic-normal distribution may be used in a fair number of areas such as economics, finance, physics and earth sciences where unimodal non-normal distributions have been empirically encountered.

A total of 10 values of $(\overline{m}_3, \overline{m}_4)$ are chosen from the region ***R***. The 10 chosen values of $(\overline{m}_3, \overline{m}_4)$ together with the corresponding values of $\boldsymbol{\lambda}$ are shown in Table (1).

For γ < 0.5, the γ-quantile Q_γ may be expressed as

$$Q = \mu + \lambda_1(-z_\gamma) + \lambda_2\left[\lambda_3(-z_\gamma)^2 - \frac{1+\lambda_3}{2}\right]$$

where z_γ is the $100(1-\gamma)\%$ point of the standard normal distribution. If we transform $(\mu, \lambda_1, \lambda_2, \lambda_3)$ to $(\beta_1, \beta_2, \beta_3, \beta_4)$ using $\beta_1 = \mu, \beta_2 = \lambda_1, \beta_3 =$

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$\lambda_2, \beta_4 = \lambda_2\lambda_3$, then Q_γ can be expressed as the following linear combination of the transformed parameters:

$$Q_\gamma = \sum_{i=1}^{4} c_i \beta_i$$

where $c_1 = 1, c_2 = -z_\gamma, c_3 = -0.5$ and $c_4 = (-z_\gamma)^2 - 0.5$.

In this section, we consider three methods for finding confidence intervals for the γ -quantile of y_i when $\gamma < 0.5$ (When $\gamma \geq 0.5$ these methods can still be applied after some slight modifications.). The first method considered is the bootstrap procedure. In this procedure, we first find the sample mean $\hat{\mu} = \bar{y}$ and fit a quadratic-normal distribution $QN(0, \hat{\boldsymbol{\lambda}})$ to the residuals

$$r_i = y_i - \hat{\mu},\ i = 1,2,\ldots,n \quad (2)$$

using the maximum likelihood procedure. The corresponding estimated γ-quantile is then given by $\hat{Q} = \sum_{i=1}^{4} c_i \hat{\beta}_i$ where $\hat{\beta}_1 = \hat{\mu}, \hat{\beta}_2 = \hat{\lambda}_1, \hat{\beta}_3 = \hat{\lambda}_2$ and $\hat{\beta}_4 = \hat{\lambda}_2\hat{\lambda}_3$. We next generate B values of $\tilde{\mathbf{y}} = (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n)^T$ using the equation $\tilde{y}_i = \hat{\mu} + \tilde{\varepsilon}_i$ where $\tilde{\varepsilon}_i \sim QN(0, \hat{\boldsymbol{\lambda}})$.

For each generated value of $\tilde{\mathbf{y}}$, we obtain $\tilde{\mu} = \sum \tilde{y}_i / n$ and fit $\tilde{y}_i - \tilde{\mu}$ with a quadratic-normal distribution $QN(0, \tilde{\boldsymbol{\lambda}})$ using the maximum likelihood procedure. Leting $\tilde{\beta}_1 = \tilde{\mu}, \tilde{\beta}_2 = \tilde{\lambda}_1$, $\tilde{\beta}_3 = \tilde{\lambda}_3$ and $\tilde{\beta}_4 = \tilde{\lambda}_2\tilde{\lambda}_3$, we next obtain the following estimate

$$\tilde{Q} = \sum_{i=1}^{4} c_i \tilde{\beta}_i$$

of the γ-quantile for the $QN(\hat{\mu}, \hat{\boldsymbol{\lambda}})$ distribution. Let the B values of $\tilde{Q}$ be denoted as $\tilde{Q}^{(1)}, \tilde{Q}^{(2)}, \ldots, \tilde{Q}^{(B)}$. Suppose $\tilde{Q}^{(1)}, \tilde{Q}^{(2)}, \ldots, \tilde{Q}^{(B)}$ can be fitted by the quadratic-normal distribution with mean $\bar{\tilde{Q}} = \frac{1}{B}\sum_{b=1}^{B} \tilde{Q}^{(b)}$ and parameter $\tilde{\boldsymbol{\lambda}}^*$. The limits of the two-sided bootstrap confidence interval for the γ-quantile are then given by the following $100(\alpha/2)\%$ and $100(1-\alpha/2)\%$ points of the $QN\left(\bar{\tilde{Q}}, \tilde{\boldsymbol{\lambda}}^*\right)$ distribution:

$$Q_L = \bar{\tilde{Q}} + \tilde{\lambda}_1^*(-z_{\alpha/2}) + \tilde{\lambda}_2^*\left[\tilde{\lambda}_3^*(-z_{\alpha/2})^2 - \frac{1+\tilde{\lambda}_3^*}{2}\right] \quad (3)$$

and

$$Q_U = \bar{\tilde{Q}} + \tilde{\lambda}_1^*(z_{\alpha/2}) + \tilde{\lambda}_2^*\left[(z_{\alpha/2})^2 - \frac{1+\tilde{\lambda}_3^*}{2}\right]$$

(4)

The limits of the two-sided bootstrap confidence interval for the γ -quantile can be modified by using Efron's BC_a method. The procedure is described below.

After obtaining B values of $\tilde{Q}$, we find the proportion $\hat{G}(\hat{Q})$ of the values $\tilde{Q}^{(1)}, \tilde{Q}^{(2)}, \cdots, \tilde{Q}^{(B)}$ which are less than $\hat{Q}$ and obtain the bias correction $z_0 = \Phi^{-1}(\hat{G}(\hat{Q}))$, $\Phi(\bullet)$ being the standard normal cumulative distribution function. Let the density of $F_{\boldsymbol{\eta}}$ be denoted by $f_{\boldsymbol{\eta}}$, where $\boldsymbol{\eta} = (\mu, \lambda_1, \lambda_2, \lambda_3)^T$. Let $\mathbf{I}_{\boldsymbol{\eta}}$ be the 4×4 matrix with ijth entry $-(\partial^2/\partial\eta_i\partial\eta_j)\log f_{\boldsymbol{\eta}}$ evaluated at $\boldsymbol{\eta} = \hat{\boldsymbol{\eta}}$. Next let ∇ be the gradient vector of Q_γ evaluated at $\hat{\boldsymbol{\eta}}$, and $\hat{\mathbf{s}} = (\mathbf{I}_{\boldsymbol{\eta}})^{-1}\nabla$. The acceleration constant a in the BC_a procedure is then given by $a = SKEW_{t=0}(d/dt(\log f_{\hat{\boldsymbol{\eta}}+t\hat{\mathbf{s}}}))/6$.

The 100(1-α)% Efron's BC_a confidence interval for the γ -quantile is given by $\left[\hat{G}^{-1}(\Phi(z(\alpha/2))), \hat{G}^{-1}(\Phi(z(1-\alpha/2)))\right]$, where $z(\alpha/2) = z_0 + (z_0 + z^{(\alpha/2)})/(1 - a(z_0 + z^{(\alpha/2)}))$ and $z^{(\alpha/2)} = \Phi^{-1}(\alpha/2)$.

We may also use the normal approximation for the distribution of $\tilde{Q}$ to construct the following confidence interval for the γ -quantile:

$$\left[\bar{\tilde{Q}} - z_{\alpha/2} S_{\tilde{Q}}, \bar{\tilde{Q}} + z_{\alpha/2} S_{\tilde{Q}}\right]$$

where $S_{\tilde{Q}} = \left[\frac{1}{B-1}\sum_{b=1}^{B}\left(\tilde{Q}^{(b)} - \bar{\tilde{Q}}\right)^2\right]^{\frac{1}{2}}$.

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The third method for constructing confidence interval for γ-quantile is based on hypothesis testing. This method is described below.

First consider the problem of testing $H_0 : Q_\gamma = Q_\gamma^{(0)}$ against $H_1 : Q_\gamma \neq Q_\gamma^{(0)}$. Suppose we test the above H_0 by using the decision rule "Accept H_0 at the α level if $Q_L^{(0)} \le \hat{Q} \le Q_U^{(0)}$" where $Q_L^{(0)}$, $Q_U^{(0)}$ are respectively the $100(\alpha/2)\%$ and $100(1-\alpha/2)\%$ points of the quadratic-normal distribution which is used to fit the B values of $\tilde{Q}$ obtained when the B values of $\tilde{\mathbf{y}}$ are generated using $\tilde{y}_i = \mu^{(m)} + \tilde{\varepsilon}_i$ where $\tilde{\varepsilon}_i \sim QN(0, \boldsymbol{\lambda}^{(m)})$ and $(\mu^{(m)}, \boldsymbol{\lambda}^{(m)})$ is the value of $(\mu, \boldsymbol{\lambda})$ which minimizes

$$D^2 = \sum_{i=1}^{4} (\beta_i - \hat{\beta}_i)^2 \quad (5)$$

subject to

$$\sum_{i=1}^{4} c_i \beta_i = Q_\gamma^{(0)}.$$

To understand the reason for choosing $(\mu^{(m)}, \lambda^{(m)})$, we may first imagine that the parameter vector $\boldsymbol{\beta}$ can be transformed to another parameter vector $(Q_\gamma, \rho_1, \rho_2, \rho_3)^T$ which consists of four linearly independent functions of $\boldsymbol{\beta}$. The last three components ρ_i, $1 \le i \le 3$ in the transformed parameter vector are known as the nuisance parameters. The value $(\mu^{(m)}, \lambda^{(m)})$ chosen by using the above procedure will correspond to a value $\boldsymbol{\beta}^{(m)}$ of $\boldsymbol{\beta}$, and $\boldsymbol{\beta}^{(m)}$ will in turn correspond to a transformed parameter vector of which the first component corresponds to a value $Q_\gamma^{(0)}$ of the γ-quantile which is different from the γ-quantile $\hat{Q}_\gamma$ based on $\hat{\boldsymbol{\beta}}$ which corresponds to $(\hat{\mu}, \hat{\boldsymbol{\lambda}})$, but both $\boldsymbol{\beta}^{(m)}$ and $\hat{\boldsymbol{\beta}}$ will have the same values of the underlying nuisance parameters. The difference between $Q_\gamma^{(0)}$ and $\hat{Q}_\gamma$ may then be assessed using the difference between $\boldsymbol{\beta}^{(m)}$ and $\hat{\boldsymbol{\beta}}$. However the components of $\tilde{\boldsymbol{\beta}}$ which corresponds to $\tilde{\boldsymbol{y}}$ generated by using $\boldsymbol{\beta}^{(m)}$ may be correlated and having different variances. This means that although $\boldsymbol{\beta}^{(m)}$ is nearest to $\hat{\boldsymbol{\beta}}$ physically, $\hat{\boldsymbol{\beta}}$ may not be nearest to $\boldsymbol{\beta}^{(m)}$ if the distance measure is given by the expected distance of $\tilde{\boldsymbol{\beta}}$ from $\hat{\boldsymbol{\beta}}$.

It would be a good strategy to make the statistical distance become more consistent with the physical distance because the decision rule used for judging whether $\boldsymbol{\beta}^{(m)}$ is close enough to $\hat{\boldsymbol{\beta}}$ is based on the distribution of $\tilde{\boldsymbol{\beta}}$. To make the two types of distance measure become more consistent with one another, we transform $\boldsymbol{\beta}$ further to $\boldsymbol{\xi}$ via $\boldsymbol{\xi} = \boldsymbol{A}\boldsymbol{\beta}$ where $\boldsymbol{A}^T\boldsymbol{A} = \hat{\boldsymbol{V}}^{-1}$ and $\hat{\boldsymbol{V}}^{-1}$ is the variance-covariance matrix of $\hat{\boldsymbol{\beta}}$. The components of the estimate $\tilde{\boldsymbol{\xi}}$ of $\boldsymbol{\xi}$ would now be approximately uncorrelated and each having unit variance. With this transformation, the distance square D^2 given by (5) should be modified to

$$\tilde{D}^2 = (\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})^T \hat{\boldsymbol{V}}^{-1} (\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}).$$

An approximately-$100(1-\alpha)\%$ confidence interval for the γ-quantile is then given by

$$\{Q_\gamma^{(0)} : \text{The null hypothesis that } Q_\gamma = Q_\gamma^{(0)} \text{ is accepted at the } \alpha \text{ level}\}.$$

The above hypothesis testing procedure based on D^2 has previously been used by Ng and Pooi (2008) and Goh, Y. L. (2008) while the one based on $\tilde{D}^2$ has been utilized by Tan, S. K. (2009) .

3. NUMERICAL RESULTS

For a value of $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)^T$ given in Table (1), we estimate the coverage probabilities of the confidence intervals in Section 2 using a simulation procedure in which $N = 1000$ values of $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$ are generated. For each generated value of $\mathbf{y}$, we find the confidence intervals using the methods in Section 2. For each method, we use the proportion of $\mathbf{y}$ (out of N values of $\mathbf{y}$) of which the corresponding confidence interval covers the true value of the 0.01-quantile to estimate the coverage probability. Columns 4, 5, 6, 7 and 8 in Table (2) give respectively the estimated coverage probabilities of the confidence intervals based on bootstrap, Efron's BC_a, normal approximation, hypothesis testing in which D^2 is used and hypothesis testing in which $\tilde{D}^2$ is used. The last five columns give respectively the average lengths of the five types of confidence interval.

Table (2) reveals that the coverage probabilities of the confidence intervals based on hypothesis testing tend to be closer to the target

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value 0.95 than those of the confidence intervals based on bootstrap, Efron's BC_a and normal approximation.

Table (2) also shows that the confidence interval based on the hypothesis testing procedure in which $\tilde{D}^2$ is used has a better coverage probability but longer average length than the confidence interval based on the hypothesis testing procedure in which D^2 is used. This is not surprising because in order to have a larger coverage probability, the length of the confidence interval should be made longer.

When the sample size n is increased from 50 to 100, improvements of the coverage probabilities of the five types of confidence interval are also observed in Table (2).

4. CONCLUDING REMARKS

When the underlying distribution is non-normal, it is not easy to use non-parametric methods to make inference regarding the γ -quantile for small values of γ (for example 0.01 and 0.001). One of the reasons is that the sample size required would then be very large.

An improvement over the non-parametric methods is given by the semi-parametric techniques.

The present paper uses a parametric approach based on the quadratic-normal distribution. This distribution is of great help in three aspects. Firstly, due to its versatility, it can be used to fit a wide class of non-normal data. Secondly, it allows the γ -quantile to be estimated from the observations $y_1, y_2, \ldots, y_n$ even when γ is small and the sample size is only moderately large. Thirdly, the process of using B bootstrap samples to estimate the (α /2)-quantile and (1- α /2)-quantile of the distribution of the estimate $\tilde{Q}$ for the γ -quantile is made simpler if we fit a quadratic-normal distribution to the B values of $\tilde{Q}$.

In using the method based on hypothesis testing for constructing confidence interval for the γ -quantile in the presence of nuisance parameters, we have made use of the value of the parameter vector which corresponds to the quantile value under the null hypothesis, and is "nearest" to the parameter vector estimated based on the observations $y_1, y_2, \ldots, y_n$. The choice of this value of the parameter vector helps us to obtain confidence interval for the γ -quantile with an improved coverage probability and a satisfactory expected length.

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Table 1.
Parameters of selected quadratic-normal distributions

No	$\overline{m}_3$	$\overline{m}_4$	λ_1	λ_2	λ_3
1	0	4	0.791976	0.127466	-0.99997
2	0	9.4	0.200746	0.468172	-1
3	0.2	7	0.413812	0.369542	-0.88671
4	0.4	3.4	0.949308	0.092855	0.376996
5	0.4	10	0.164777	0.52339	-0.86164
6	0.6	8.8	0.274307	0.483371	-0.76038
7	0.8	5.8	0.61955	0.32209	-0.39263
8	1.4	13	0.051082	0.662814	-0.63272
9	1.6	8.8	0.479657	0.474738	-0.2262
10	2	10.6	0.433434	0.535119	-0.14424

Table 2.
Estimated coverage probabilities and average lengths of confidence intervals for γ -quantile
(γ =0.01, α =0.05, μ =0, N=1000, B=5000)

n	$\overline{m}_3$	$\overline{m}_4$	Estimated coverage probability					Average length				
			Bootstrap	Efron's BC_a	Normal Approximation	Hypothesis testing (D^2)	Hypothesis testing ($\tilde{D}^2$)	Bootstrap	Efron's BC_a	Normal Approximation	Hypothesis testing (D^2)	Hypothesis testing ($\tilde{D}^2$)
50	0	4	0.803	0.851	0.785	0.893	0.915	1.81008	1.9693387	1.81407	2.874007	3.248347
	0	9.4	0.679	0.812	0.642	0.862	0.903	2.52505	3.047406	2.53685	4.080545	4.707265
	0.2	7	0.723	0.826	0.688	0.88	0.906	2.18651	2.5701492	2.19466	3.543256	4.132106
	0.4	3.4	0.814	0.818	0.803	0.884	0.906	1.14972	1.2108826	1.1457	1.918769	2.042169
	0.4	10	0.71	0.848	0.666	0.86	0.89	2.46811	3.02193	2.47938	4.030156	4.658056
	0.6	8.8	0.711	0.832	0.672	0.882	0.904	2.20028	2.6779962	2.20888	3.611796	4.300496
	0.8	5.8	0.713	0.802	0.689	0.855	0.867	1.44279	1.6795123	1.44339	2.425541	2.837851
	1.4	13	0.663	0.825	0.618	0.85	0.887	2.05853	2.6925457	2.06608	3.469681	4.177391
	1.6	8.8	0.683	0.826	0.653	0.863	0.889	1.16555	1.4967658	1.15945	2.076065	2.404885
	2	10.6	0.729	0.843	0.678	0.859	0.887	0.99967	1.3392289	0.98836	1.815295	2.072505
100	0	4	0.852	0.891	0.843	0.91	0.929	1.41991	1.5296974	1.4185	1.844542	2.174572
	0	9.4	0.806	0.897	0.771	0.877	0.906	2.23425	2.6391247	2.23199	2.939191	3.332251
	0.2	7	0.786	0.894	0.754	0.914	0.935	1.82017	2.1123367	1.8161	2.39872	2.81848
	0.4	3.4	0.848	0.851	0.838	0.886	0.907	0.85788	0.8942856	0.85777	1.167221	1.335401
	0.4	10	0.736	0.887	0.7	0.863	0.897	2.01904	2.4591742	2.01291	2.683725	3.096805
	0.6	8.8	0.737	0.873	0.706	0.888	0.916	1.8063	2.1933967	1.79929	2.419622	2.837142
	0.8	5.8	0.74	0.844	0.71	0.871	0.915	1.12634	1.2821194	1.1215	1.558242	1.866942
	1.4	13	0.668	0.843	0.619	0.854	0.888	1.71576	2.2894652	1.70411	2.392309	2.821899
	1.6	8.8	0.691	0.85	0.654	0.885	0.917	0.90913	1.1670927	0.89314	1.401608	1.683128
	2	10.6	0.73	0.861	0.68	0.882	0.912	0.75854	1.0120045	0.73865	1.254416	1.482026

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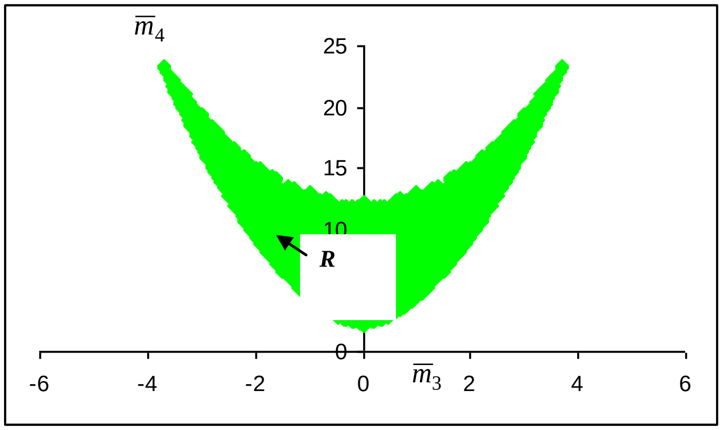

Figure 1. Possible values of $(\overline{m}_3, \overline{m}_4)$ such that ε_i is a one-to-one function of e_i.