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ON A COMPUTATIONAL METHOD FOR SOLVING MIXED INTEGRAL EQUATION WITH SINGULAR KERNEL

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ABSTRACT:

*In this paper, under certain conditions, the existence of a unique solution of mixed integral equation of the second kind (**MIESK**) with singular kernel is considered and proved in the space* $L_2(\Omega)\times C[0,T],T<1$, *using Banach fixed point theorem. Then, using a numerical method, we have a system of Fredholm integral equations (**SFIEs**). Moreover we use Toeplitz matrix method (**TMM**) to obtain a linear algebraic system (**LAS**) which can be solved numerically. The error estimate is discussed. Some examples are solved numerically.*

Keywords: Mixed integral equation, system of Fredholm integral equations, linear algebraic system, singular kernel, Toeplitz matrix method.

1. INTRODUCTION:

The theory of integral equations with its application has close contact with many different areas of different sciences. These different problems have led researches to establish different methods for solving integral equations of different types and different kinds with continuous or discontinuous kernel. Many authors have interested in solving the linear and nonlinear integral equations. In [1], Brunner et al. introduced a class of methods depending on some parameters for the numerical solution of Abel integral equation of

$$\mu\varphi(x,t)=f(x,t)+\lambda\int_0^t\int_\Omega k(|x-y|)F(t,\tau)\phi(y,\tau)dyd\tau+\lambda\int_\Omega k(|x-y|)\phi(y,\tau)\,dy$$

in the space $L_2[\Omega]\times C[0,T]$, where f,k,F are known functions. $k(|x-y|)$ is called the kernel of Fredholm integral term, in position, and $F(t,\tau)$ is called the kernel of Volterra integral term, in time. The constant μ defines the kind of the integral equation, while the constant λ, may be complex, and has a physical meaning.

For this, we write Eq. (1) in the integral form

$$\overline{W}\phi(x,t)=\frac{1}{\mu}f(x,t)+\frac{\lambda}{\mu}W\phi(x,t),\quad(\mu\neq 0),\quad (2)$$

$$W\phi(x,t)=\int_0^t\int_\Omega F(t,\tau)k(|x-y|)\phi(y,\tau)dy\,d\tau+\int_\Omega k(|x-y|)\phi(y,t)dy$$

In order to guarantee the existence of a unique solution, we assume the following

(i) The kernel of position satisfies the discontinuity condition

$$\left\{\int_\Omega\int_\Omega |k(|x-y|)|^2\,dx\,dy\right\}^{\frac{1}{2}}=M$$

(M is a constant).

(ii) The kernel of Volterra integral term belongs to the class and satisfies, for the constant, the condition

(iii) The given function with its partial derivatives with respect to position and time are continuous in and its norm is defined as

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International Journal of Engineering and Applied Sciences

www.eaas-journal.org

$$\|f(x,t)\|_{L_2(\Omega)\times C[0,T]} = \max_{0\le t\le T}\left|\int_0^t\left\{\int_\Omega |f(x,\tau)|^2 dx\right\}^{\frac{1}{2}} d\tau\right| = H,$$

(H is a constant)

Theorem (1) without proof: The **MIE** (1) has a unique solution in the space $L_2(\Omega)\times C[0,T]$ under the condition $|\mu| > 2|\lambda|TNM\,; T = \max_{0\le t\le\tau} t$.

In the remainder part of this paper, we use a numerical method, to obtain **SFIEs**, where the existence and uniqueness of the solution can be discussed and proved. Then, we use the **TMM** to represent the **SFIEs** as **LAS** which can be solved numerically. Some examples are considered, when the kernel takes two forms of singularity. Also the error, in each case is estimated.

2. System of Fredholm integral equations: In this section, a numerical method for Eq. (1), is used to obtain **SFIEs** in position. For this aim, we divide the interval $[0,T]$, as $0 = t_0 < t_1 < \ldots < t_1 < \ldots < t_N = T$, where $t = t_k$, $k = 0,1,2\ldots,N$. Hence, the integral term of Eq. (.1) becomes

$$\int_0^{t_\ell}\int_\Omega F(t_\ell,\tau)k(|x-y|)\phi(y,\tau)dy\,d\tau = \sum_{j=0}^{\ell} u_j F(t_\ell,t_j)\int_\Omega k(|x-y|)\phi(y,t_j)dy + R \quad (3)$$

The order of R are depending on the number of the derivatives of $F(t,\tau)$ for all $t,\tau\in[0,T]$ with respect to t, h denotes the constant step size for integration, u_j are the weights, such that

$$u_j = \begin{cases} h/2 & j=0,\ j=\ell \\ h & 0<j<\ell_; \end{cases} \quad ,\ell = 0,1,2\ldots,p, \quad (4)$$

Using (3) in (1) we have the following **SFIEs** of the second kind

$$\mu\phi_\ell(x) = f_\ell(x) + \lambda\sum_{j=0}^{\ell} u_j F_{\ell,j}\int_\Omega k(|x-y|)\phi_j(y)dy + \lambda\int_\Omega k(|x-y|)\phi_j(y)\,dy + R \quad (5)$$

Here, after using the following notation: $\phi_\ell(x) = \phi(x,t_\ell), F_{\ell,j} = F(t_\ell,t_j), f_\ell(x) = f(x,t_\ell)$, $(0\le j\le \ell)$, and neglecting the error R, the formula (5) takes the form:

$$\mu_\ell\,\phi_\ell(x) - \lambda(u_\ell F_{\ell,j}+1)\int_\Omega k(|x-y|)\phi_\ell(y)dy = H_\ell(x), \quad (6)$$

$$H_\ell(x) = f_\ell(x) + \lambda\sum_{j=0}^{\ell-1}(u_j F_{\ell,j}+1)\int_\Omega k(|x-y|)\phi_j(y)dy\,]; \qquad \mu_\ell = \mu - \lambda u_\ell F_{\ell,\ell}.$$

The solution of **SFIEs** of Eq. (6) can be obtained, using the recurrence relations.

3. The Toeplitz matrix method (see Abdou et al. [13, 14]): Here, we will discuss the solution of Eq. (1) or Eq. (5) numerically, using **TMM** in one dimensional, and $\Omega = [-b,b]$, we have

$$\int_{-b}^{b} k(|x-y|)\phi(y)dy = \sum_{n=-N}^{N} D_n^{(j)}(x)\phi_j(nh), \quad (7)$$

International Journal of Engineering and Applied Sciences

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$$D_n^{(j)}(x)=\begin{cases}A_{-N}^{(j)}(x), & n=-N\\ A_n^{(j)}(x)+B_{n-1}^{(j)}(x), & -N<n<N\\ B_{N-1}^{(j)}(x), & n=N\ .\end{cases}$$

Where,

$$A_n^{(j)}(x)=\frac{1}{h}\left[(a+h)\,I(x)-J(x)\ \right],\quad B_n^{(j)}(x)=\frac{1}{h}\left[\,J(x)-aI(x)\right]. \tag{8}$$

$$I^{(j)}(x)=\int_a^{a+h}k(|x-y|)\,dy\ ,\quad J^{(j)}(x)=\int_a^{a+h}k(|x-y|)\,y\,dy\,,$$

Thus, the integral equation (6) after Putting $x=mh$ and using the following notations:

$\phi_i(\ell h)=\phi_{i\ell},\ D_n^{(i)}(mh)=D_{mn}^{(i)},\ f_i(mh)=f_{im},\ \phi_j(nh)=\phi_{jn},\ D_n^{(j)}(mh)=D_{mn}^{(j)}$,

Takes the form

$$\mu_\ell\,\phi_{\ell m}-\lambda\sum_{n=-N}^{N}(u_\ell F_{\ell\ell}+1)D_{mn}^{(\ell)}\,\phi_{\ell n}=H_{\ell m}\ ,\quad -N\le m\le N \tag{9}$$

$$H_{m\ell}=f_{m\ell}+\lambda\Big(\sum_{j=0}^{\ell-1}\sum_{n=-N}^{N}u_j\,F_{\ell,j}+1\Big)D_{mn}^{(j)}\phi_{jn}\,;\quad \mu_\ell=\mu-\lambda u_\ell\,F_{\ell,\ell}$$

The matrices $D_{mn}^{(j)}$ can be written in the Toeplitz matrices form

$D_{mn}^{(j)}=G_{mn}^{(j)}-E_{mn}^{(j)},\quad G_{mn}^{(j)}=A_n^{(j)}(mh)+B_{n-1}^{(j)}(mh),\ -N\le m,n\le N\,,$

$$E_{mn}^{(j)}=\begin{cases}B_{-N-1}^{(j)}(mh), & n=-N\\ 0 & -N<n<N\\ A_N^{(j)}(mh), & n=N\ ,\ j=1,2,\dots,i\end{cases}$$

Here, the matrix $G_{mn}^{(j)}$ is called Toeplitz matrix of order $(2N+1)X\,(2N+1)$ and $E_{mn}^{(j)}$ represent matrices of order $(2N+1)X\,(2N+1)$ whose elements are zeros except the first and the last rows (columns).

The error term $R^{(j)}$ can be determined from the following formula

$$R^{(j)}=\max_j\left|\int_{nh}^{nh+h}y^2k\,(|x-y|)dy-[A_n^{(j)}(x)(nh)^2+B_n^{(j)}(x)(nh+h)^2]\right|=O\,(h^3) \tag{10}$$

4. The existence and uniqueness of the solution of LAS: This section will be devoted, to prove the existence of a unique solution of the **LAS** (9) in the Banach space ℓ^∞. For this, we write it in the operator form

International Journal of Engineering and Applied Sciences

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$$\bar{T}\phi_m = T\phi_m + \frac{1}{\mu} f_m, \tag{11}$$

$$T\phi_{im} = \frac{\lambda}{\mu}\sum_{j=0}^{i}\sum_{n=-N}^{N}(u_j F_{ij}+1)D_{mn}^{(j)}\phi_{nj}; \quad (\mu \neq 0, -N \le m \le N)$$

Then, we consider the following

Lemma (1): If the kernel of Eq. (1) satisfies the following conditions

$$k(|x-y|) \in L_2[-b,b]; \quad \lim_{x'\to x}\|k(|x'-y|)-k(|x-y|)\| = 0; x, x' \in [-b,b], \tag{12}$$

Then, for $\sup_i \sum_{j=0}^{i}|F_{ij}| < q_1, \quad q_1 \neq 0$ we have

(i) $\sup_N \sup_i \sum_{j=0}^{i}\sum_{n=-N}^{N}|F_{ij}||D_{mn}^{(j)}| \le E, \quad E = \max_j E^{(j)}, \quad ;(E$ is a constant)

(ii) $\lim_{m'\to m}\sup_i \sup_N \sum_{j=0}^{i}\sum_{n=-N}^{N}|F_{ij}||D_{m'n}^{(j)} - D_{mn}^{(j)}| = 0.$

Proof: For $A_n^{(j)}(x)$ of the formula (8), after applying Hölder inequality, and summing from $n=-N$ to $n=N$, we have

$$\sum_{n=-N}^{N}|A_n^{(j)}(x)| \le \frac{1}{|h|}\sum_{j=0}^{i}|F_{ij}|\|k(|x-y|)\|\left[\sum_{n=-N}^{N}|a+h|\|1\|+|1|\|y\|\right]$$

In view of the first condition (i), there exists a small constant E_1, such that

$\sum_{n=-N}^{N}|A_n^{(j)}(x)| \le E_1, \quad \forall N, \forall j = 0,1,2,...,i, \quad E_1 = \max_j E^{(j)}$

Since, each term of $\sum_{n=-N}^{N}A_n^{(j)}(x)$ is bounded above, hence for $x = mh$ we deduce that

$$\sup_j \sup_N \sum_{n=-N}^{N}|A_n^{(j)}(mh)| \le E_1$$

Similarly, for $B_n^{(j)}(x)$ of (8), we can find a small constant E_2, such that

$$\sup_j \sup_N \sum_{n=-N}^{N}|B_n^{(j)}(mh)| \le E_2, \quad E_2 = \max_j E_2^{(j)}.$$

In the of the above results, there exists a small constant E , such that

$$\sup_j \sup_N \sum_{j=0}^{i}\sum_{n=-N}^{N}|F_{ij}||D_{mn}^{(j)}| \le \sup_j \sup_N \sum_{n=-N}^{N}|A_n^{(j)}(mh)| + \sup_j \sup_N \sum_{n=-N}^{N}|B_n^{(j)}(mh)| \le E \; ; (E = E_1 + E_2)$$

Hence, case (i) of lemma (1) is satisfied.■

By virtue of the formula (8), we get for $x, x' \in [-b,b]$

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$$\left|A_n^{(j)}(x')-A_n^{(j)}(x)\right| \le \frac{1}{|h|}\left\{\sum_{j=0}^{i}\left|F_{ij}\right|\left\{\left|a+h\right|\int_a^{a+h}\left|k(|x'-y|)-k(|x-y|)\right|\left|1\right|dy \right.\right.$$

$$\left.\left. +\left|1\right|\int_a^{a+h}\left|k(|x'-y|)-k(|x-y|)\right|\left|y\right|dy\right\}\right\}$$

Applying Hölder inequality, then summing from $n=-N$ to $n=N$, the above inequality can be adapted in the form

$$\lim_{m'\to m}\sup_j\sup_N\sum_{n=-N}^{N}\left|A_n^{(j)}(m'h)-A_n^{(j)}(mh)\right|=0.$$

Similarly, we can prove $\lim_{m'\to m}\sup_j\sup_N\sum_{n=-N}^{N}\left|B_n^{(j)}(m'h)-B_n^{(j)}(mh)\right|=0.$

Finally, case (ii) of lemma (1) is proved satisfied.■

Now, the existence of a unique solution of the **LAS** (9), can be proved according to the Banach fixed point theorem. For this, consider the following assumptions

$$\sup_m\left|f_m\right|\le H<\infty, \quad (H \text{ is a constant}) \tag{13}$$

$$\sup_j\sup_N\sum_{j=0}^{i}\sum_{n=-N}^{N}\left|F_{ij}\right|\left|D_{mn}^{(j)}\right|\le E^*, \quad E^*=\max_j E^{*(j)} \quad ;(E^* \text{is a constant}) \tag{14}$$

For the constants $Q>Q_1$, $Q>P_1$; the points $\phi(n\,h)$, satisfies

$$\sup_n\left|\phi(n\,h)\right|\le Q_1\left\|\Phi\right\|_{\ell^\infty}, \quad \sup_n\left|\phi(n\,h)-\psi(n\,h)\right|\le P_1\left\|\Phi-\Psi\right\|_{\ell^\infty}, \tag{15}$$

Theorem (3): The **LAS** of Eq. (9), in the space ℓ_∞ has a unique solution under the condition

$\left|\lambda\right|(QE^*+M)<\left|\mu\right|.$

To prove this theorem, we must consider the following lemmas.

Lemma (2): If the conditions (13) – (15) are verified, then the operator $\bar{T}$ defined by Eq. (11) maps the space ℓ^∞ into itself.

Proof: Let U be the set of all continuous functions $\Phi=\{\phi_m\}$ in ℓ^∞ such that $\left\|\Phi\right\|_{\ell^\infty}\le\beta$, β is a constant.

Define the norm of the operator $\bar{T}\Phi$ in the Banach space ℓ^∞ by $\left\|\bar{T}\Phi\right\|_{\ell^\infty}=\sup_m\left|\bar{T}\phi_m\right|$, for each integer m.

After using the conditions (13) and (14), we get

$$\left|\bar{T}\phi_m\right|\le\left|\frac{\lambda}{\mu}\right|Q\left\|\Phi\right\|_{\ell^\infty}\sup_j\sup_N\sum_{j=0}^{i}\sum_{n=-N}^{N}\left(\left|F_{ij}\right|\left|D_{mn}^{(j)}\right|+\left|D_{mn}^{(j)}\right|\right)+\frac{H}{|\mu|},(0\le j\le i) \tag{16}$$

In view of condition (14), the above inequality, for each integer m can be adapted in the form

$$\left\|\bar{T}\Phi\right\|_{\ell^\infty}\le\sigma_1\left\|\Phi\right\|_{\ell^\infty}+\frac{H}{|\mu|}, \quad \sigma_1=\left|\frac{\lambda}{\mu}\right|(QE^*+M). \tag{17}$$

The inequality (17) shows that, the operator $\bar{T}$ maps the set U into itself, where $\left|\mu\right|\beta(1-\sigma_1)=H$. Since $\beta>0$, $H>0$,

International Journal of Engineering and Applied Sciences

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therefore we have $\sigma_1 < 1$. Also, the inequality of Eq. (17) involves the boundedness of the operator T, furthermore the boundedness of the operator $\bar{T}$. ■

Lemma (3): Under the two conditions (13) and (15), $\bar{T}$ is continuous and a contraction operator in the space ℓ^∞.

Proof: For the two functions Φ and Ψ in ℓ^∞, the operator $\bar{T}$ leads to

$$\left|\bar{T}\phi_m - \bar{T}\psi_m\right| \le \left|\frac{\lambda}{\mu}\right| Q \left\|\Phi - \Psi\right\|_{\ell^\infty} \sup_j \sup_N \left(\sum_{j=0}^{i} \left|F_{ij}\right| + 1\right) \sum_{n=-N}^{N} \left|D_{mn}^{(j)}\right|.$$

Using the condition (15), we get $\left\|\bar{T}\Phi - \bar{T}\Psi\right\| \le \sigma_1 \left\|\Phi - \Psi\right\|$, this inequality shows that, the operator $\bar{T}$ is continuous in the space ℓ^∞. Moreover $\bar{T}$ is a contraction operator, under the condition $\sigma_1 < 1$.■

In the light of the lemmas (2) and (3), the operator $\bar{T}$ is a contraction in the Banach space ℓ^∞. Hence, the **LAS** has a unique solution in ℓ^∞■

In the next theorem, the convergence of sequence of approximate solution to the exact solution of **LAS** of Eq. (9) will be proved in the space ℓ^∞.

Theorem (4): If the conditions (13) - (15) are satisfied and the sequence of functions $\{L_j\} = \{(f_m)_j\}$ converges uniformly to the function $L = \{f_m\}$ in the space ℓ^∞. Then, the sequence of approximate solution $\{\Phi_j\} = \{(\phi_m)_j\}$ converges uniformly to the solution $\Phi = \{\phi_m\}$ of Eq. (9) in ℓ^∞.

Proof: By virtue of Eq. (9), we have

$$\left|\phi_m - (\phi_m)_j\right| \le \left|\frac{\lambda}{\mu}\right| \left(\left|\sum_{j=0}^{i} h \left|F_{ij}\right| \sum_{n=-N}^{N} \left|D_{mn}\right| + 1\right) \sup_n \left|\phi(nh) - \phi_j(nh)\right| + \frac{1}{|\mu|} \sup_n \left|f_m - (f_m)_j\right|.$$

The above inequality, after using the condition (16) for each integer m, we get

$$\left\|\Phi - \Phi_j\right\|_{\ell^\infty} \le \frac{1}{[|\mu| - |\lambda|(E^* Q + M)]} \left\|L - L_j\right\|_{\ell^\infty}; \quad (\sigma_1 < 1) \tag{18}$$

Since $\left\|L - L_j\right\|_{\ell^\infty} \to 0$ as $j \to \infty$, so that $\left\|\Phi - \Phi_j\right\|_{\ell^\infty} \to 0$■

When $N \to \infty$, it is natural to expect the sum

$$\sum_{j=0}^{i} \sum_{n=-N}^{N} F_{ij} D_{mn}^{(j)} \phi_{nj} = \int_0^t \int_{-b}^{b} F(t,\tau) k(x,y) \phi(y,\tau) dy d\tau; \quad -N \le m \le N.$$

Hence, the solution of the **LAS** (9) becomes a solution of the **IE** (1).

Theorem (5): If the sequence of continuous functions $\{f_j(x,t)\}$ converges uniformly to the function $f(x,t)$ in the space $L_2[\Omega] \times C[0,T]$., Then, under the conditions of theorem (2), the sequence of the approximate solution $\{\phi_j(x,t)\}$ converges uniformly to the exact solution of Eq. (1) in the space $L_2[\Omega] \times C[0,T]$.

International Journal of Engineering and Applied Sciences

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Proof: The formula (1) with its approximate solution in the space $L_2[\Omega]\times C[0,T]$ and for $\mu \neq 0$ gives

$$\left\|\phi(x,t)-\phi_j(x,t)\right\| \leq \left|\frac{\lambda}{\mu}\right| \left\| \int_0^t \int_\Omega \left|F(t,\tau)k(|x-y|)\right|.\left|\phi(y,\tau)-\phi_j(y,\tau)\right| dy d\tau + \int_\Omega \left|k(|x-y|)\right| \left|\phi(y,\tau)-\phi_j(y,\tau)\right| dy \right\|$$

$$+\frac{1}{\mu}\left\|f(x,t)-f_j(x,t)\right\|$$

Using the conditions (i) – (ii), and applying Hölder inequality, we get

$$\left\|\phi(x,t)-\phi_j(x,t)\right\| \leq \left(\frac{1}{|\mu|-|\lambda|(E^{*}Q+M)}\right)\left\|f(x,t)-f_j(x,t)\right\|. \ .$$

Finally, we have

$$\left\|\phi(x,t)-\phi_j(x,t)\right\| \rightarrow 0 \text{ , since } \left\|f(x,t)-f_j(x,t)\right\| \rightarrow 0 \text{ as } j \rightarrow \infty. \blacksquare$$

In any numerical practical, we need to have the size of the error involved. Fortunately, the following two definitions enable us to calculate the error of the method.

Definition (1): The estimate local error R_j is determined by the following

$$R_j = \left| \int_0^t \int_\Omega F(t,\tau)k(x-y)\phi(y,\tau)dy d\tau + \int_\Omega k(x-y)\phi(y,\tau)dy - \sum_{j=0}^{i} w_j F_{ij} \sum_{n=-N}^{N} D_{mn}^{(j)}\phi(nh,\tau_i) - \sum_{n=-N}^{N} D_{mn}^{(j)}\phi(nh,\tau_i) \right|.$$

Definition (2): The **TMM** is said to be convergent of order r in the interval $[-b,b]$, if and only if for sufficiently large N , there exists a constant $D>0$ independent of N such that

$$\left\|\phi(x,t)-\phi_N(x,t)\right\| \leq D\ N^{-r}.$$

5. Applications and Concoction

Application 1: Consider the generalized logarithmic kernel

$k(|x-y|)=(\ln|x-y|)^q, \quad q=1,2,\ldots,n$. For the following example

$$\mu\varphi(x,t)-\lambda\int_0^t\int_{-1}^{1} \tau^2.(\ln|x-y|)^q.\varphi(y,\tau)dy d\tau - \lambda\int_{-1}^{1}(\ln|x-y|)^q.\varphi(y,t)dy = f(x,t);$$

$$\mu=1,\ N=40,\quad \lambda=0.001, q=2. \quad (\varphi(x,t)=x^2t^2)$$

In our example application we use the result of the following integral

$$\int x^n(\ln x)^m dx = \frac{x^{m+1}}{m+1}\sum_{J=0}^{m}(-1)^j\,(m+1)m(m-1)\ldots(m-J+1)\frac{(\ln x)^{m-J}}{(n+1)^{J+1}}, \quad (n\neq -1)$$

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www.eaas-journal.org

t	ϕ_E	ϕ_T	E_T
$t=0.1$	0.04000000000	0.040069433291	$6.9433290\text{x}10^{-5}$
	0.04400797894	0.044084369406	$7.6390470\text{x}10^{-5}$
	0.05650736349	0.056605450796	$9.8087305\text{x}10^{-5}$
	0.07696786148	0.077101464774	$1.3360330\text{x}10^{-4}$
	0.09882869040	0.099000240433	$1.7155003\text{x}10^{-4}$
$t=0.4$	0.09000000000	0.090197978690	$1.9797869\text{x}10^{-4}$
	0.09901795261	0.099235768655	$2.1781605\text{x}10^{-4}$
	0.12714156785	0.127421249198	$2.7968134\text{x}10^{-4}$
	0.17317768832	0.173558638231	$3.8094991\text{x}10^{-4}$
	0.22236455341	0.222853702774	$4.8914937\text{x}10^{-4}$
$t=0.8$	0.49000000000	0.488091707686	$1.9082923\text{x}10^{-3}$
	0.53909774196	0.536998239764	$2.0995022\text{x}10^{-3}$
	0.69221520276	0.689519388574	$2.6958142\text{x}10^{-3}$
	0.94285630386	0.939184373624	$3.6719295\text{x}10^{-3}$
	0.12106514575	1.20593660669	$4.7148508\text{x}10^{-3}$

Table (1)

Application 2: Consider the **MIE** with Hilbert kernel in the form

$$\mu\varphi(x,t)-\lambda\int_{0}^{t}\int_{-\pi}^{\pi}t^{2}\tau.\cot\left(\frac{y-x}{2}\right)\varphi\ (y,\tau)dyd\tau-\lambda\int_{-\pi}^{\pi}\cot\left(\frac{y-x}{2}\right).\varphi(y,t)dy=f\ (x,t);$$

$\mu=1, N=40,\quad \lambda=0.001.\quad (\varphi(x,t)=t^{2}\sin^{2}x,\ )$

t	ϕ_E	ϕ_T	E_T
$t=0.1$	0.04000000000	0.04007508731	$7.5087305\text{x}10^{-5}$
	0.04277127585	0.04285156534	$8.0289495\text{x}10^{-5}$
	0.05136101667	0.05145743067	$9.6414008\text{x}10^{-5}$
	0.06594885083	0.06607264887	$1.2379804\text{x}10^{-4}$
	0.08468000067	0.08483896049	$1.5895983\text{x}10^{-4}$
$t=0.4$	0.09000000000	0.09023446612	$2.3446612\text{x}10^{-4}$
	0.09623537066	0.09648608104	$2.5071038\text{x}10^{-4}$
	0.11556228750	0.11586334796	$3.0106046\text{x}10^{-4}$
	0.14838491436	0.14877148365	$3.8656928\text{x}10^{-4}$
	0.19053000150	0.19102636628	$4.9636478\text{x}10^{-4}$
$t=0.8$	0.16000000000	0.16048420328	$4.8420328\text{x}10^{-4}$
	0.17108510339	0.17160285319	$5.1774980\text{x}10^{-4}$
	0.20544406670	0.20606579599	$6.2172932\text{x}10^{-4}$

International Journal of Engineering and Applied Sciences

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	0.26379540331	0.26459371956	$7.9831625x10^{-4}$
	0.33872000266	0.33974506101	$1.0250584x10^{-3}$

Table (2)

Application (3): Consider the **MIE** with Carleman function form

$$\mu\varphi(x,t)-\lambda\int_0^t\int_{-1}^1 t^2\tau.|x-y|^{-\nu}\varphi(y,\tau)dyd\tau-\lambda\int_{-1}^1|x-y|^{-\nu}.\varphi(y,\tau)d\tau=f(x,t);$$

$\mu=1, N=40,\ \nu=0.39, \lambda=0.001.\quad (\varphi(x,t)=x^2t^2)$

t	ϕ_E	ϕ_T	E_T
$t=0.1$	0.040000000000	0.040079245943	$7.9245943x10^{-5}$
	0.041224516651	0.041306188544	$8.1671893x10^{-5}$
	0.044963778011	0.045052857936	$8.9079925x10^{-5}$
	0.051361016668	0.051462770473	$1.0175381x10^{-4}$
	0.060464436471	0.060584225504	$1.1978903x10^{-4}$
$t=0.4$	0.090000000000	0.090261312446	$2.6131245x10^{-4}$
	0.092755162465	0.093024474447	$2.6931198x10^{-4}$
	0.101168500525	0.101462240396	$2.9373987x10^{-4}$
	0.115562287502	0.115897819325	$3.3553182x10^{-4}$
	0.136044982061	0.136439984806	$3.9500275x10^{-4}$
$t=0.8$	0.160000000000	0.160591577021	$5.9157702x10^{-4}$
	0.1648980666042	0.165507753523	$6.0968692x10^{-4}$
	0.179855112045	0.180520100491	$6.6498845x10^{-4}$
	0.205444066670	0.206203666601	$7.5959993x10^{-4}$
	0.241857745886	0.242751980166	$8.9423428x10^{-4}$

Table (3)

From Tables (1),(2) and (3) and our results, we can conclude the following

(i)As N increases the estimate error, in linear and nonlinear case is decreases. While the time increases, the error is increasing, also.

(ii)In Table (3) as ν is increasing, ν is called Poisson's ratio, the error is increasing. The importance of Carleman kernel come from the work of Arutinuion [15], who has shown that the plane contact problem in the nonlinear theory of plasticity , in its first approximation can be reduced to Fredholm integral equation of the first or the second kind according to the conditions of problem .

(iii)The **TMM** is considered as one of the best method for solving the singular integral equation, where the singularity disappears and the solution can be obtained directly see Abdou et al. [13,14].

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